3D flows of Newtonian and viscoplastic fluids with free surface

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Outline

- Models for Newtonian and viscoplastic fluid one-phase free-surface flow
- Level set method for free surface capturing
- Numerical scheme
- Verification tests
- Examples of 3D flows

A model for free-surface viscous fluid flow

Fluid domain: $\Omega(t) \in \mathbb{R}^3$ with boundary $\overline{\partial\Omega(t)} = \overline{\Gamma_D} \cup \overline{\Gamma(t)}$
Γ_D: solid part, $\Gamma(t)$: free surface

Fluid equations:

$$\begin{cases} \rho\left(\dfrac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u}\right) - \mathbf{div}\,\boldsymbol{\tau} + \nabla p = \mathbf{f} \\ \nabla\cdot\mathbf{u} = 0 \end{cases} \quad \text{in } \Omega(t),$$

Newtonian fluid constitutive law

$$\boldsymbol{\tau} = \mu\mathbf{D}\mathbf{u} \quad \mathbf{D}\mathbf{u} = \nabla\mathbf{u} + (\nabla\mathbf{u})^T$$

μ: viscosity parameter,
ρ: density of fluid, $\mathbf{u}$: velocity vector, p: kinematic pressure,
$\mathbf{Du}$: rate of strain tensor,
$\boldsymbol{\tau}$: deviatoric part of the stress tensor

A model for free-surface viscoplastic fluid flow

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The Herschel-Bulkley constitutive law

$$\begin{aligned} \boldsymbol{\tau} = \left(K\,|\mathbf{Du}|^{n-1} + \tau_s|\mathbf{Du}|^{-1}\right)\mathbf{Du} \;&\Leftrightarrow\; |\boldsymbol{\tau}| > \tau_s, \\ \mathbf{Du} = \mathbf{0} \;&\Leftrightarrow\; |\boldsymbol{\tau}| \le \tau_s. \end{aligned}$$

$K > 0$: consistency parameter, $\tau_s \ge 0$: yield stress parameter, $n > 0$: flow index,
ρ: density of fluid, $\mathbf{u}$: velocity vector, p: kinematic pressure,
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Note: Mathematically sound formulations are written in terms of variational inequalities (Duvaut, Lions 1976).

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Fluid equations (regularization):

$$\left\{ \begin{aligned} \rho\left(\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u}\right) - \mathbf{div}\,\mu_\varepsilon \mathbf{Du} + \nabla p = \mathbf{f} \\ \nabla\cdot\mathbf{u} = 0 \end{aligned} \right. \quad \text{in } \Omega(t),$$

with the shear-dependent effective viscosity

$$\mu_\varepsilon = K\,|\mathbf{Du}|_\varepsilon^{n-1} + \tau_s |\mathbf{Du}|_\varepsilon^{-1}, \qquad |\mathbf{Du}|_\varepsilon = \sqrt{|\mathbf{Du}|^2 + \varepsilon^2}.$$

$K > 0$: consistency parameter, $\tau_s \geq 0$: yield stress parameter, $n > 0$: flow index,
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Modeling error:

$$\|\mathbf{u}_0 - \mathbf{u}_\varepsilon\|_{H^1} \leq \sqrt{\varepsilon}$$

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$$\mu_\varepsilon = K\,|\mathbf{D}\mathbf{u}|_\varepsilon^{n-1} + \tau_s |\mathbf{D}\mathbf{u}|_\varepsilon^{-1}, \qquad |\mathbf{D}\mathbf{u}|_\varepsilon = \sqrt{|\mathbf{D}\mathbf{u}|^2 + \varepsilon^2}.$$

Initial and boundary conditions:

$$\Omega(0) = \Omega_0, \quad \mathbf{u}|_{t=0} = \mathbf{u}_0 \quad \text{and} \quad \mathbf{u} = \mathbf{g} \quad \text{on } \Gamma_D.$$

$K > 0$: consistency parameter, $\tau_s \geq 0$: yield stress parameter, $n > 0$: flow index,
ρ: density of fluid, $\mathbf{u}$: velocity vector, p: kinematic pressure,
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Balance of the surface tension and stress forces:

$$(\mu_\varepsilon \mathbf{D}\mathbf{u} - p\,\mathbf{I})\mathbf{n}_\Gamma = \varsigma\kappa\mathbf{n}_\Gamma - p_{\text{ext}}\mathbf{n}_\Gamma \quad \text{on } \Gamma(t),$$

and kinematic condition on $\Gamma(t)$

$$v_\Gamma = \mathbf{u}|_\Gamma \cdot \mathbf{n}_\Gamma.$$

$K > 0$: consistency param., $\tau_s \geq 0$: yield stress param., $n > 0$: flow index,
ρ: density of fluid, $\mathbf{u}$: velocity vector, p: kinematic pressure,
$\mathbf{Du}$: rate of strain tensor, ε: regularization param., $\mathbf{n}_\Gamma$: normal vector for $\Gamma(t)$,
v_Γ: normal velocity of $\Gamma(t)$, ς: surface tension coef., κ: sum of principal curvature

Interface capturing: Level set approach

Idea:(Sethian, Osher '87)

$\Gamma(t)$ = zero-level of a scalar function

The level set function $\varphi(x,t)$

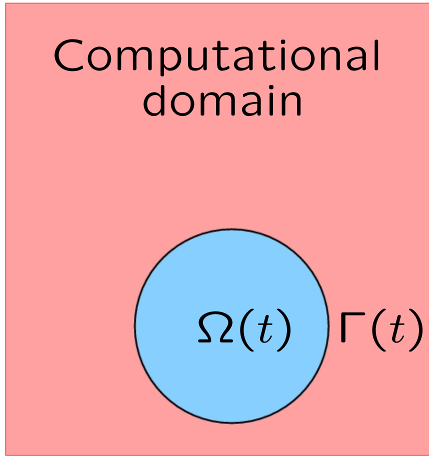
$$\phi(x,t) = \begin{cases} < 0 & \text{for } x \text{ in fluid domain } \Omega(t) \\ > 0 & \text{for } x \text{ in } \mathbf{R}^3 \setminus \Omega(t) \\ = 0 & \text{at the free surface} \end{cases}$$

should be an
"approximate signed distance function".

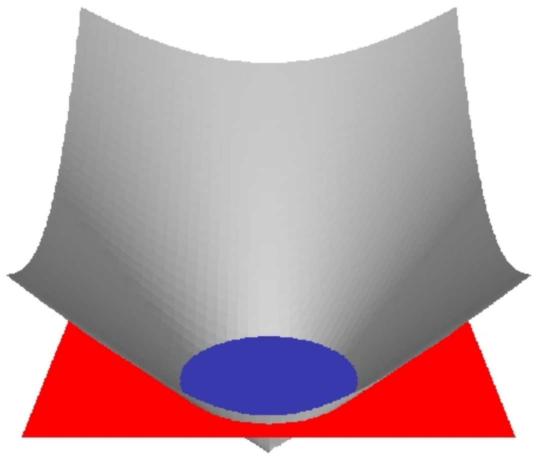
$$x(t) \in \Gamma(t) \quad \Rightarrow \quad \phi(x(t),t) = 0.$$

Level set equation

$$\boxed{\phi_t + \widetilde{\mathbf{u}} \cdot \nabla\phi \quad = \quad 0 \quad \text{in} \quad \mathbb{R}^3}$$



$\Downarrow$



A model for free-surface viscoplastic fluid flow

Fluid domain: $\Omega(t) \in \mathbb{R}^3$ with boundary $\overline{\partial\Omega(t)} = \overline{\Gamma_D} \cup \overline{\Gamma(t)}$
Γ_D: solid part, $\Gamma(t)$: free surface

Fluid + level set equations + b.c. + i.c. (coupling between fluid and level set eqs. are in red):

$$\begin{cases} \rho\left(\dfrac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u}\right) - \mathbf{div}\, \mu_\varepsilon \mathbf{D u} + \nabla p = \mathbf{f} \\ \nabla \cdot \mathbf{u} = 0 \\ \mu_\varepsilon = K\,|\mathbf{Du}|_\varepsilon^{n-1} + \tau_s |\mathbf{Du}|_\varepsilon^{-1} \end{cases} \quad \text{in } \Omega(t),$$

$$\mathbf{u}|_{t=0} = \mathbf{u}_0 \quad \text{and} \quad \mathbf{u} = \mathbf{g} \quad \text{on } \Gamma_D, \quad (\mu_\varepsilon \mathbf{Du} - p\,\mathbf{I})\mathbf{n}_\Gamma = \varsigma \kappa \mathbf{n}_\Gamma \quad \text{on } \Gamma(t)$$

$$\begin{cases} \dfrac{\partial \phi}{\partial t} + \widetilde{\mathbf{u}} \cdot \nabla \phi = 0 \quad \text{in } \mathbb{R}^3 \times (0, T] \\ \phi(0) = \phi_0, \end{cases}$$

with $\mathbf{n}_\Gamma = \nabla\phi/|\nabla\phi|$, and $\kappa = \nabla \cdot \mathbf{n}_\Gamma$.

Distance property: $|\nabla \phi| = 1$.

$K > 0$: consistency param., $\tau_s \geq 0$: yield stress param., $n > 0$: flow index,
ρ: density of fluid, $\mathbf{u}$: velocity vector, p: kinematic pressure,
$\mathbf{Du}$: rate of strain tensor, ε: regularization param., $\mathbf{n}_\Gamma$: normal vector for $\Gamma(t)$,
v_Γ: normal velocity of $\Gamma(t)$, ς: surface tension coef., κ: sum of principal curvature

A model for free-surface Newtonian fluid flow

Fluid domain: $\Omega(t) \in \mathbb{R}^3$ with boundary $\overline{\partial\Omega(t)} = \overline{\Gamma_D} \cup \overline{\Gamma(t)}$
Γ_D: solid part, $\Gamma(t)$: free surface

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$$\begin{cases} \rho\left(\dfrac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u}\right) - \mathbf{div}\,\mu\mathbf{D}\mathbf{u} + \nabla p = \mathbf{f} \\ \nabla\cdot\mathbf{u} = 0 \end{cases} \quad \text{in } \Omega(t),$$

$$\mathbf{u}|_{t=0} = \mathbf{u}_0 \quad \text{and} \quad \mathbf{u} = \mathbf{g} \quad \text{on } \Gamma_D, \quad (\mu\mathbf{D}\mathbf{u} - p\,\mathbf{I})\mathbf{n}_\Gamma = \varsigma\kappa\mathbf{n}_\Gamma \quad \text{on } \Gamma(t)$$

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Fundamentals for free-surface Newtonian fluid flow, $\rho = 1$

- Energy balance:

$$\frac{1}{2}\int_{\Omega(t)} |\mathbf{u}(t)|^2 \mathrm{d}\mathbf{x} + \int_0^t \int_{\Omega(t)} \mu |\mathbf{D}\mathbf{u}|^2 \,\mathrm{d}\mathbf{x}\,\mathrm{d}t' + \varsigma|\Gamma(t)| = \frac{1}{2}\int_{\Omega(t)} |\mathbf{u}(0)|^2 \mathrm{d}\mathbf{x} + \int_0^t \int_{\Omega(t)} \mathbf{f}\,\mathbf{u}\,\mathrm{d}\mathbf{x}\,\mathrm{d}t' + \varsigma|\Gamma(0)|,$$

  here $|\Gamma(t)| = meas_{R^2}(\Gamma(t))$.

- Momentum conservation:

$$\int_{\Omega(t)} \mathbf{u}(t)\,\mathrm{d}\mathbf{x} = \int_{\Omega(0)} \mathbf{u}(0)\,\mathrm{d}\mathbf{x} + \int_0^t \int_{\Omega(t)} \mathbf{f}\,\mathrm{d}\mathbf{x}\,\mathrm{d}t'.$$

- Angular momentum conservation:

$$\int_{\Omega(t)} \mathbf{u}(t)\times\mathbf{x}\,\mathrm{d}\mathbf{x} = \int_{\Omega(0)} \mathbf{u}(0)\times\mathbf{x}\,\mathrm{d}\mathbf{x} + \int_0^t \int_{\Omega(t)} \mathbf{f}\times\mathbf{x}\,\mathrm{d}\mathbf{x}\,\mathrm{d}t'.$$

- Mass conservation.

- Volume conservation.

K.Nikitin et al. *A splitting method for numerical simulation of free surface flows of incompressible fluids with surface tension.* Comput.Methods Appl.Math., 2015, V.15, No.1, p.59-78

Fundamentals for free-surface viscoplastic fluid flow, $\rho = 1$

- Energy inequality:

$$\frac{1}{2}\int_{\Omega(t)} |\mathbf{u}(t)|^2 \mathrm{d}\mathbf{x} + \int_0^t \int_{\Omega(t)} K|\mathbf{Du}|^{1+n} + \tau_s|\mathbf{Du}|\,\mathrm{d}\mathbf{x}\,\mathrm{d}t' + \varsigma|\Gamma(t)| \\ \leq \frac{1}{2}\int_{\Omega(t)} |\mathbf{u}(0)|^2 \mathrm{d}\mathbf{x} + \int_0^t \int_{\Omega(t)} \mathbf{f}\,\mathbf{u}\,\mathrm{d}\mathbf{x}\,\mathrm{d}t' + \varsigma|\Gamma(0)|,$$

here $|\Gamma(t)| = meas_{R^2}(\Gamma(t))$.

Note: This becomes energy <u>equality</u> (energy balance) for $\varepsilon > 0$, with $\int_0^t \int_{\Omega(t)} \mu_\varepsilon |\mathbf{Du}|^2$ standing for the dissipation term.

- Plug and yield regions.

Numerical method

Loop:

1. *Level set part*: $\Omega(t) \to \Omega(t + \Delta t)$
2. *Remeshing*
3. *Re-interpolation*
4. *Fluid part*: $\{\mathbf{u}(t), p(t)\} \to \{\mathbf{u}(t + \Delta t), p(t + \Delta t)\}$

end of the loop.

Numerical method

Loop:

1. *Level set part*: $\Omega(t) \to \Omega(t+\Delta t)$
   (a) Extend the velocity along normals to $\Gamma(t)$, $\mathbf{u}(t)|_{\Omega(t)} \to \widetilde{\mathbf{u}}(t)|_{\mathbb{R}^3}$:
   $$\mathbf{y}^0 = \mathbf{x}, \quad \mathbf{y}^{n+1} = \mathbf{y}^n - \alpha\phi_h(\mathbf{y}^n)\nabla\phi_h(\mathbf{y}^n), \quad \text{until } |\mathbf{y}^{n+1} - \mathbf{y}^n| \leq \varepsilon$$
   set $u_h(\mathbf{x}) = u_h(\mathbf{y}^{n+1})$.
   (b) Semi-Lagrangian step for $\frac{\partial\phi}{\partial t} + \widetilde{\mathbf{u}} \cdot \nabla\phi = 0$
   (c) Volume correction: Solve for δ: meas$\{\mathbf{x} : \phi(\mathbf{x}) < \delta\} = Vol^{\text{reference}}$ and correct $\phi^{new} = \phi - \delta$
   (d) Update ϕ to satisfy $|\nabla\phi| = 1$: Invokes The Marching Cubes method (Lorensen & Cline, 1987)
2. *Remeshing*
3. *Re-interpolation*
4. *Fluid part*: $\{\mathbf{u}(t), p(t)\} \to \{\mathbf{u}(t+\Delta t), p(t+\Delta t)\}$

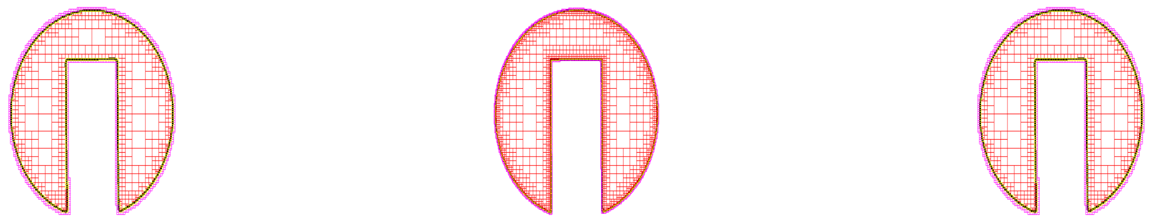
end of the loop.

Loop:

1. *Level set part*: $\Omega(t) \to \Omega(t + \Delta t)$
   (a) Extend the velocity along normals to $\Gamma(t)$, $\mathbf{u}(t)|_{\Omega(t)} \to \widetilde{\mathbf{u}}(t)|_{\mathbb{R}^3}$

   (b) Semi-Lagrangian step for $\frac{\partial \phi}{\partial t} + \widetilde{\mathbf{u}} \cdot \nabla \phi = 0$

   Zalesak's test: advection by a prescribed velocity field
   2-nd order semi-Lagrangian and enhanced with particle-level set



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2. *Remeshing*
3. *Re-interpolation*
4. *Fluid part*: $\{\mathbf{u}(t), p(t)\} \to \{\mathbf{u}(t + \Delta t), p(t + \Delta t)\}$

end of the loop.

Numerical method

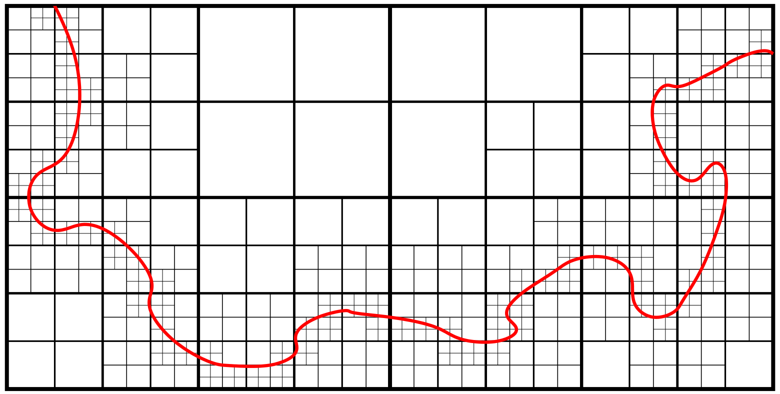
Loop:

1. *Level set part*: $\Omega(t) \rightarrow \Omega(t + \Delta t)$

2. *Remeshing*:

   (a) Graded octree cartesian mesh gradely adapted to $\Gamma(t + \Delta t)$ location.

   (b) 2D Illustration:



3. *Re-interpolation*

4. *Fluid part*: $\{\mathbf{u}(t), p(t)\} \rightarrow \{\mathbf{u}(t + \Delta t), p(t + \Delta t)\}$

end of the loop.

Numerical method

Loop:

1. *Level set part*: $\Omega(t) \to \Omega(t + \Delta t)$

2. *Remeshing*

3. *Re-interpolation*

   (a) trilinear interpolation in cubic cells

   (b) Semi-Lagrangian methods and upwind differences also use higher order interpolation

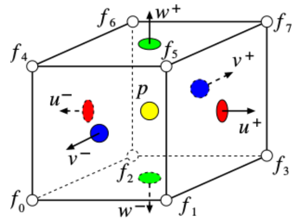
4. *Fluid part*: $\{\mathbf{u}(t), p(t)\} \to \{\mathbf{u}(t + \Delta t), p(t + \Delta t)\}$

end of the loop.

Numerical method

Loop:

1. *Level set part*: $\Omega(t) \to \Omega(t + \Delta t)$
2. *Remeshing*
3. *Re-interpolation*
4. *Fluid part*: $\{\mathbf{u}(t), p(t)\} \to \{\mathbf{u}(t + \Delta t), p(t + \Delta t)\}$



   (a) Staggered location of pressure-velocity nodes

   (b) Chorin-Yanenko type splitting:
   - Semi-Lagrangian meth. for advection and explicit visco-plastic step
   - or Solve advection-diffusion for velocity $\widetilde{\mathbf{u}}$
   - Curvature evaluation $\kappa = \nabla \cdot \nabla\phi / |\nabla\phi|$
   - Standard projection (pressure-correction) step by the solution of Poisson equation with

$$p(t + \Delta t) = \varsigma\kappa(t + \Delta t) + p_{\text{ext}} \quad \text{on } \Gamma(t + \Delta t)$$

end of the loop.

Fundamentals of <u>semi-discrete</u> scheme for Newtonian fluid, $\rho = 1$

Assumptions: backward Euler, semi-Lagrangian step for $\mathbf{u}$, C^3-smooth $\Gamma(t)$

- Momentum conservation $\int_{\Omega(t)} \mathbf{f}\,\mathrm{d}\mathbf{x} = 0$:

$$\int_{\Omega_n} \mathbf{u}^n\,\mathrm{d}\mathbf{x} = \int_{\Omega_0} \mathbf{u}^0\,\mathrm{d}\mathbf{x}.$$

- Angular momentum conservation $\int_{\Omega(t)} \mathbf{f}\times\mathbf{x}\,\mathrm{d}\mathbf{x} = 0$:

$$\int_{\Omega_n} \mathbf{u}^n \times \mathbf{x}\,\mathrm{d}\mathbf{x} = \int_{\Omega_0} \mathbf{u}^0 \times \mathbf{x}\,\mathrm{d}\mathbf{x}.$$

- Mass conservation.

- Volume conservation.

- Energy stability bound:

$$\|\mathbf{u}^M\|^2_{\Omega_M} + \sum_{n=1}^{M} \mu\Delta t\|\mathbf{D}\mathbf{u}^n\|^2_{\Omega_n} \le C\,\varsigma + \|\mathbf{u}^0\|^2_{\Omega_0} + c\sum_{n=1}^{M} \Delta t\|\mathbf{f}\|^2_{\Omega_n}$$

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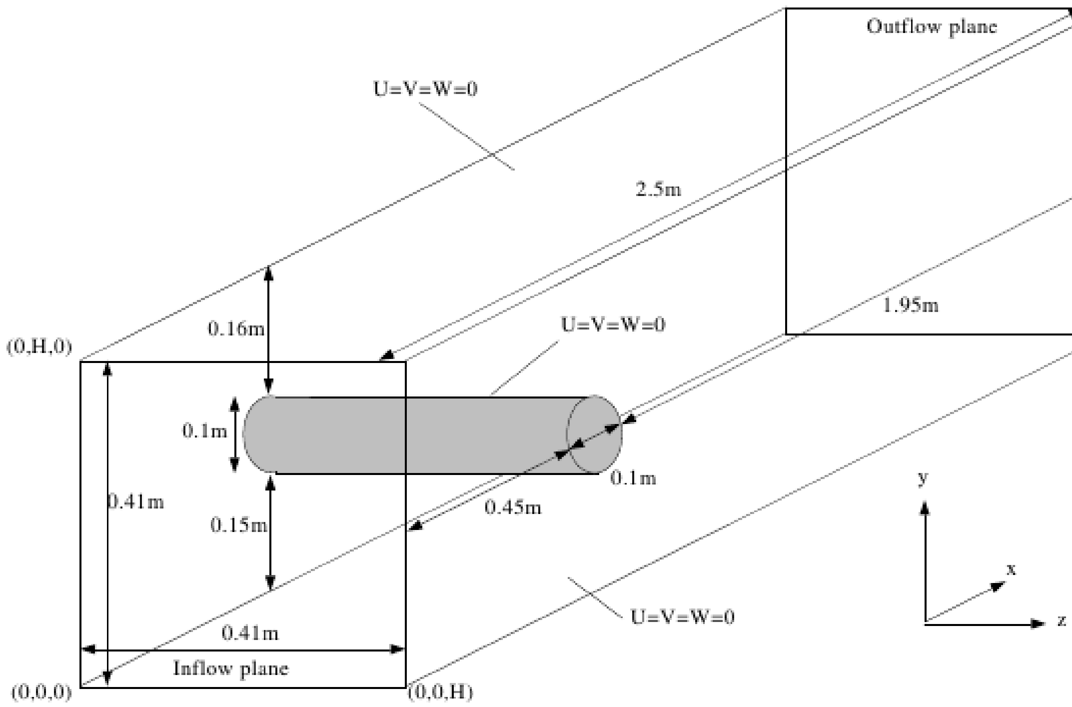
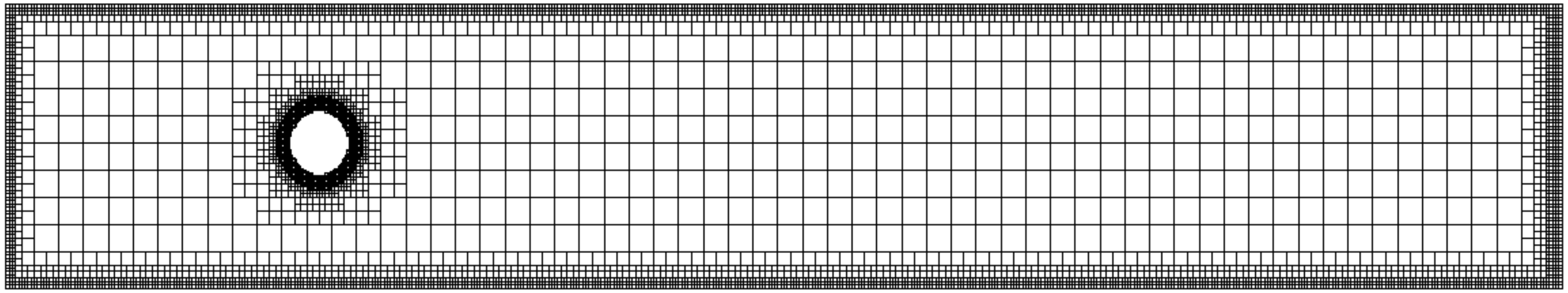
3D flow around a cylinder, Re=20



- on walls $\mathbf{u} = 0$
- on inlet $\mathbf{u} = (0, 0, 16\widetilde{U}xy(H-x)(H-y)/H^4)^T$ on Γ_{in}, $\widetilde{U} = 0.45 ms^{-1}$
- on outlet $\mu \frac{\partial \mathbf{u}}{\partial \mathbf{n}} - p\mathbf{n}|_{\Gamma_{\text{out}}} = 0$, $\mu = 10^{-3} m^2 s^{-1}$

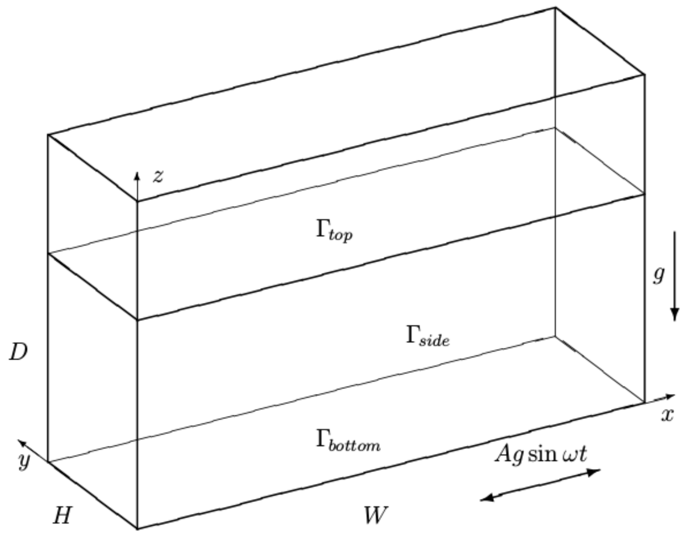
3D flow around a cylinder, Re=20



h_{min}	h_{max}	C_{drag}	C_{lift}	Δp
$\ell/128$	$\ell/64$	3.07235	-0.019821	0.13840
$\ell/256$	$\ell/64$	6.20151	0.00778	0.15961
$\ell/512$	$\ell/64$	6.15078	0.00962	0.16298
$\ell/1024$	$\ell/64$	6.14193	0.00990	0.16636
Braack & Richter		6.18533	0.009401	
Schäfer & Turek		6.05–6.25	0.008–0.01	0.165–0.175

$$C_{drag} = \frac{2}{DH\widetilde{U}^2}\int_S \left(\mu \frac{\partial(\mathbf{u}\cdot\mathbf{t})}{\partial \mathbf{n}} n_x - p n_z\right) ds \quad C_{lift} = -\frac{2}{DH\widetilde{U}^2}\int_S \left(\mu \frac{\partial(\mathbf{u}\cdot\mathbf{t})}{\partial \mathbf{n}} n_z + p n_x\right) ds$$

Sloshing tank



Value	dimenshioned	adimensioned
Lengths	$D = 0.3$ m	$\tilde{D} = 1.0$
	$H = 0.1$ m	$\tilde{H} = 0.3333$
	$W = 0.8$ m	$\tilde{W} = 2.6667$
Frequency	$f = 0.89$ s^{-1}	$\tilde{f} = 0.156$
Gravity acc.	$g = 9.81$ ms^{-2}	$\tilde{g} = 1.0$
Viscosity	$\nu = 1.0 \times 10^{-6}$ m^{2}s^{-1}	$\tilde{\nu} = 1.943 \times 10^{-6}$

- Length of tank W and horizontal excitation period are chosen to generate the mode with $\lambda = 2W$
- After 10 periods excitation is turned off
- On walls we impose slip b.c.
- We measure heights of waves on walls $h_{left}(t), h_{right}(t)$
- We compare free surfaces after 1 period of excitation

Sloshing tank

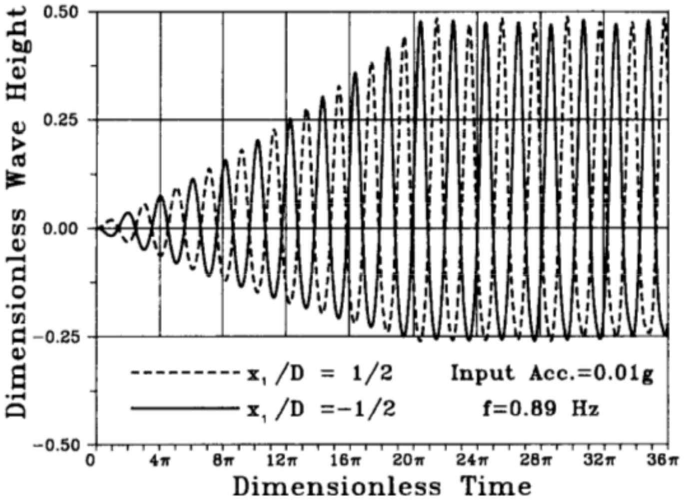
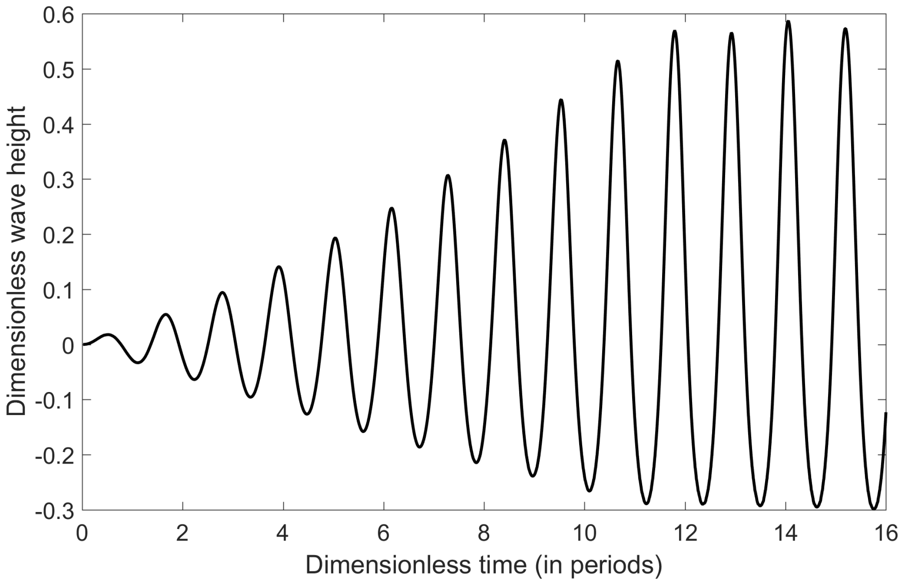


Fig. 27. Time histories of the wave heights at the wall.

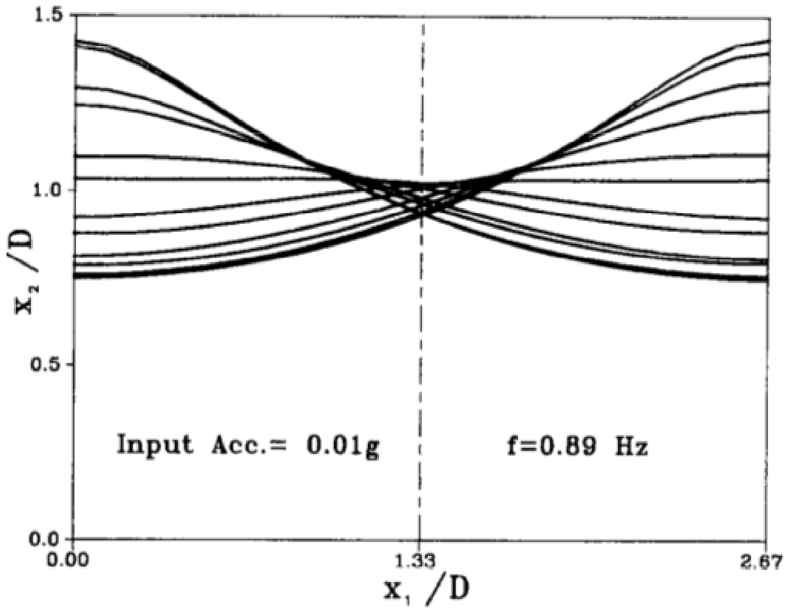


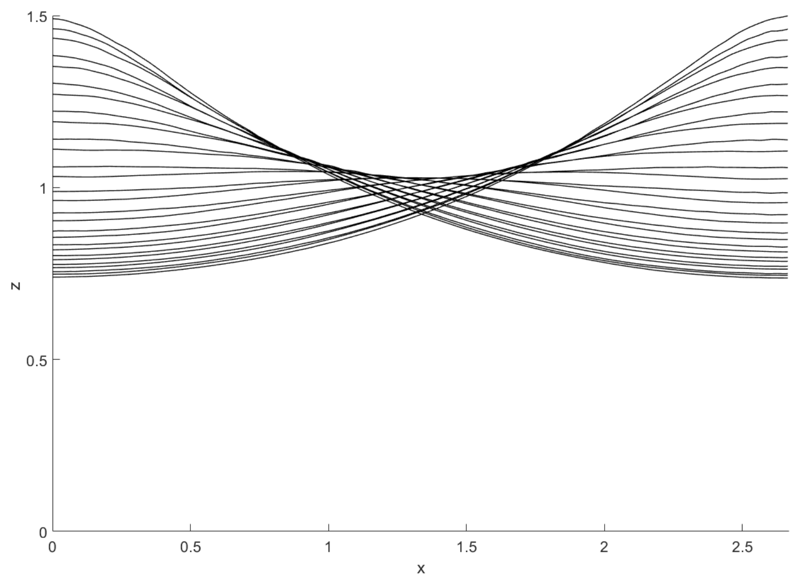
Reference heights

Computed heights

A. Huerta and W. Liu. Viscous flow with large free surface motion

Sloshing tank





Reference free surface after 1 period   Computed free surface after 1 period

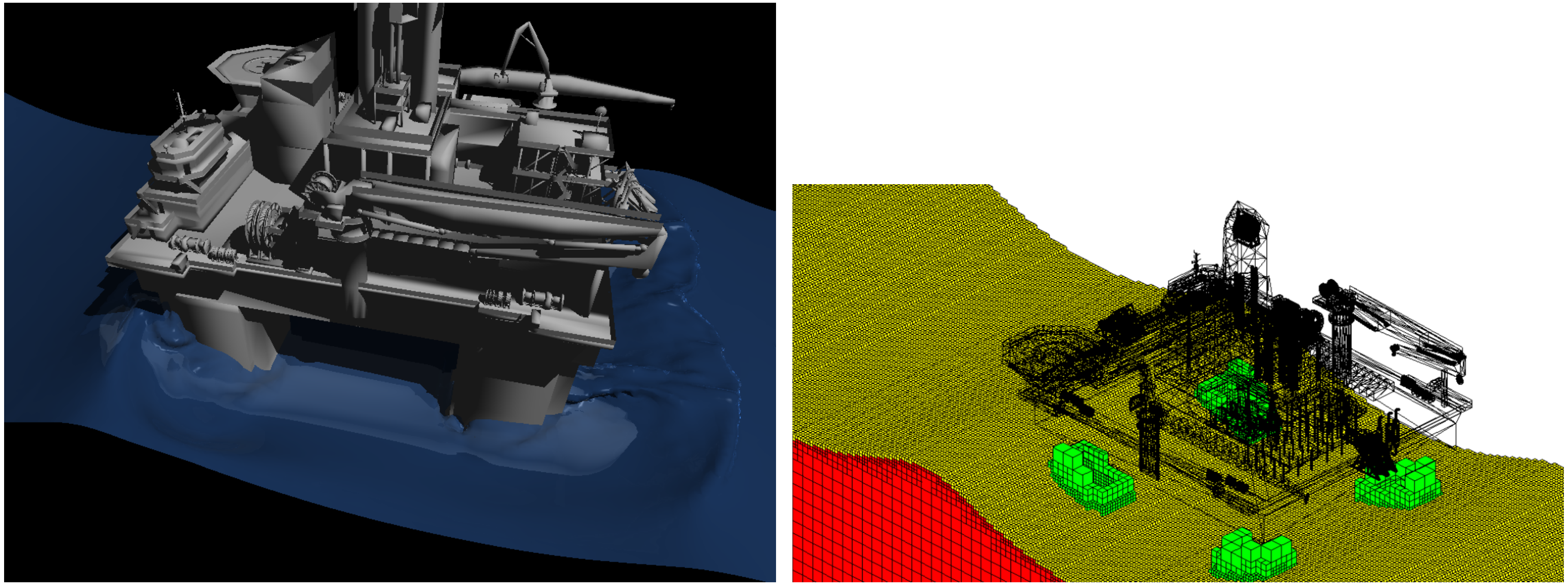
A. Huerta and W. Liu. Viscous flow with large free surface motion

Free surface flow passing offshore oil platform

- Bulk computational domain: 440m$\times$110m$\times$110m.
- The sea depth 55m
- Wave length $\lambda = 110$m, height $A = 11.5$m, period $T = 8.4$s (maximal for Kara sea)
- Inlet $x = 0$ and outlet boundaries $x = 440$m have prescribed Dirichlet b.c.
- Other boundaries (except free surface) have slip b.c.
- Compute highest water levels at the platform and dynamic forces experienced by the construction

K.Nikitin et al. *An adaptive numerical method for free surface flows passing rigidly mounted obstacles*. Computers & Fluids, 2017, V.148, 56-68

Free surface flow passing offshore oil platform



An operating offshore unit and the mesh for wave runup simulation

K.Nikitin et al. *An adaptive numerical method for free surface flows passing rigidly mounted obstacles.* Computers & Fluids, 2017, V.148, 56-68

Velocities at inlet and outlet are given by the 3d order Stokes waves

$$\mathbf{u}_{\text{wave}}(x,y,z,t) = (u(x,z,t), 0, w(x,z,t))^T \quad \text{for} \quad z \le \eta_{2D}(x,y,t),$$

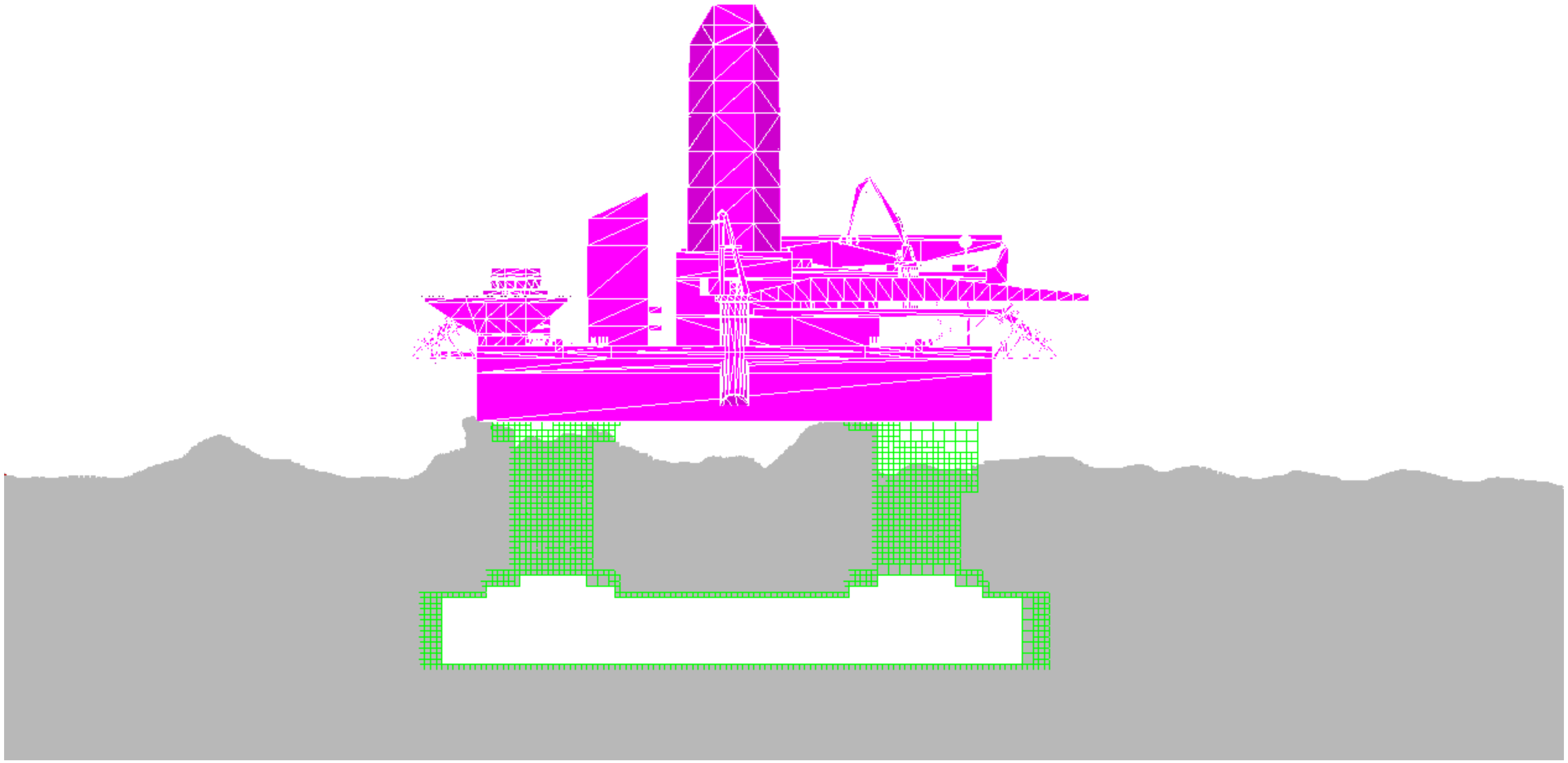
where $\eta_{2D}(x,y,t) = \eta(x,t)$ is the free surface level.

The 3d order Stokes waves are defined by superposition of the 1st order waves:

$$\begin{aligned} \eta(x,t) &= A\cos(kx-\omega t) \\ u(x,z,t) &= A\omega e^{-kz}\cos(kx-\omega t) \\ w(x,z,t) &= A\omega e^{-kz}\sin(kx-\omega t). \end{aligned}$$

K.Nikitin et al. *An adaptive numerical method for free surface flows passing rigidly mounted obstacles.* Computers & Fluids, 2017, V.148, 56-68

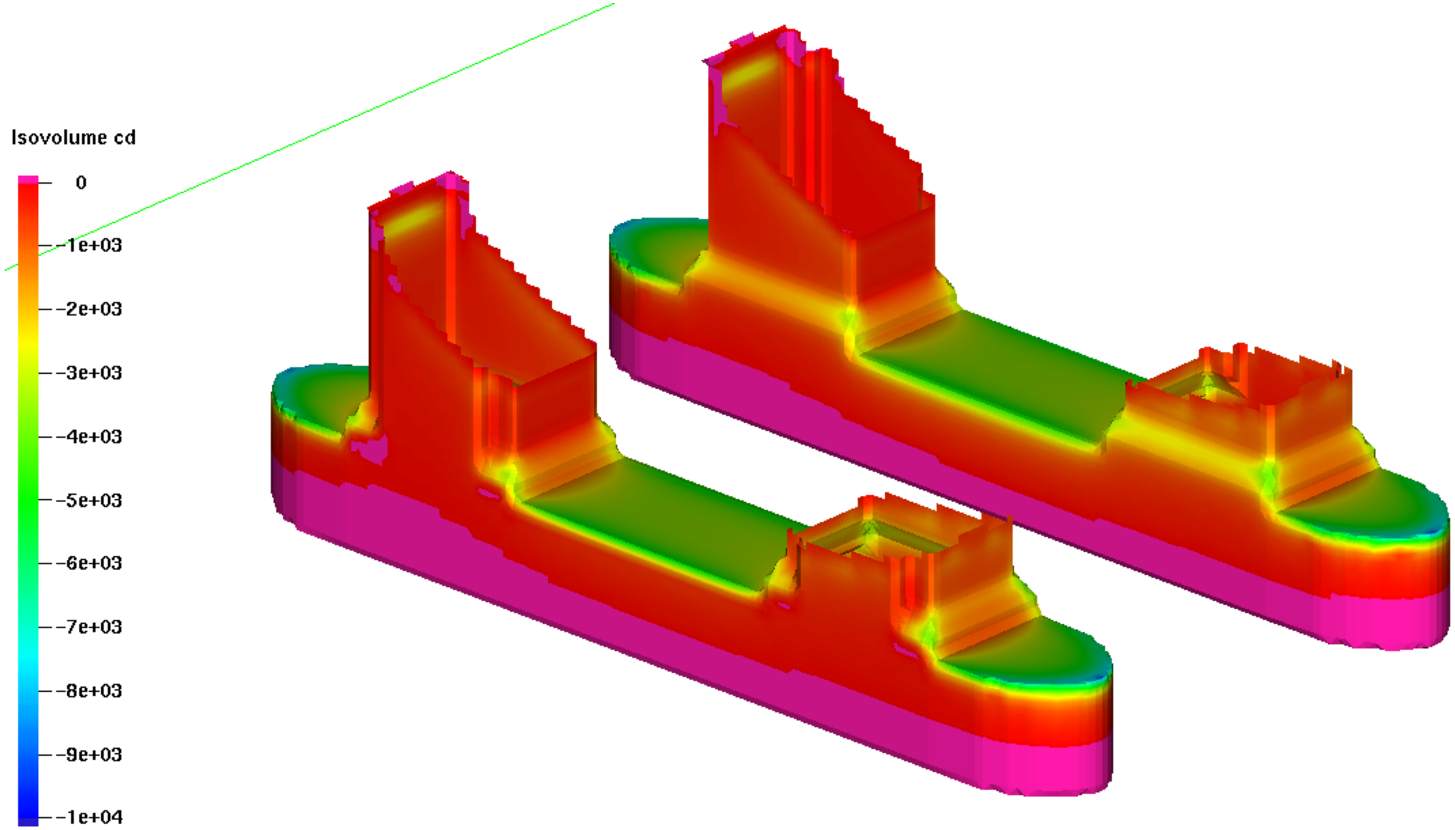
Free surface flow passing offshore oil platform



Maximum observed water level, central cross-section of the computational domain

K.Nikitin et al. *An adaptive numerical method for free surface flows passing rigidly mounted obstacles*. Computers & Fluids, 2017, V.148, 56-68

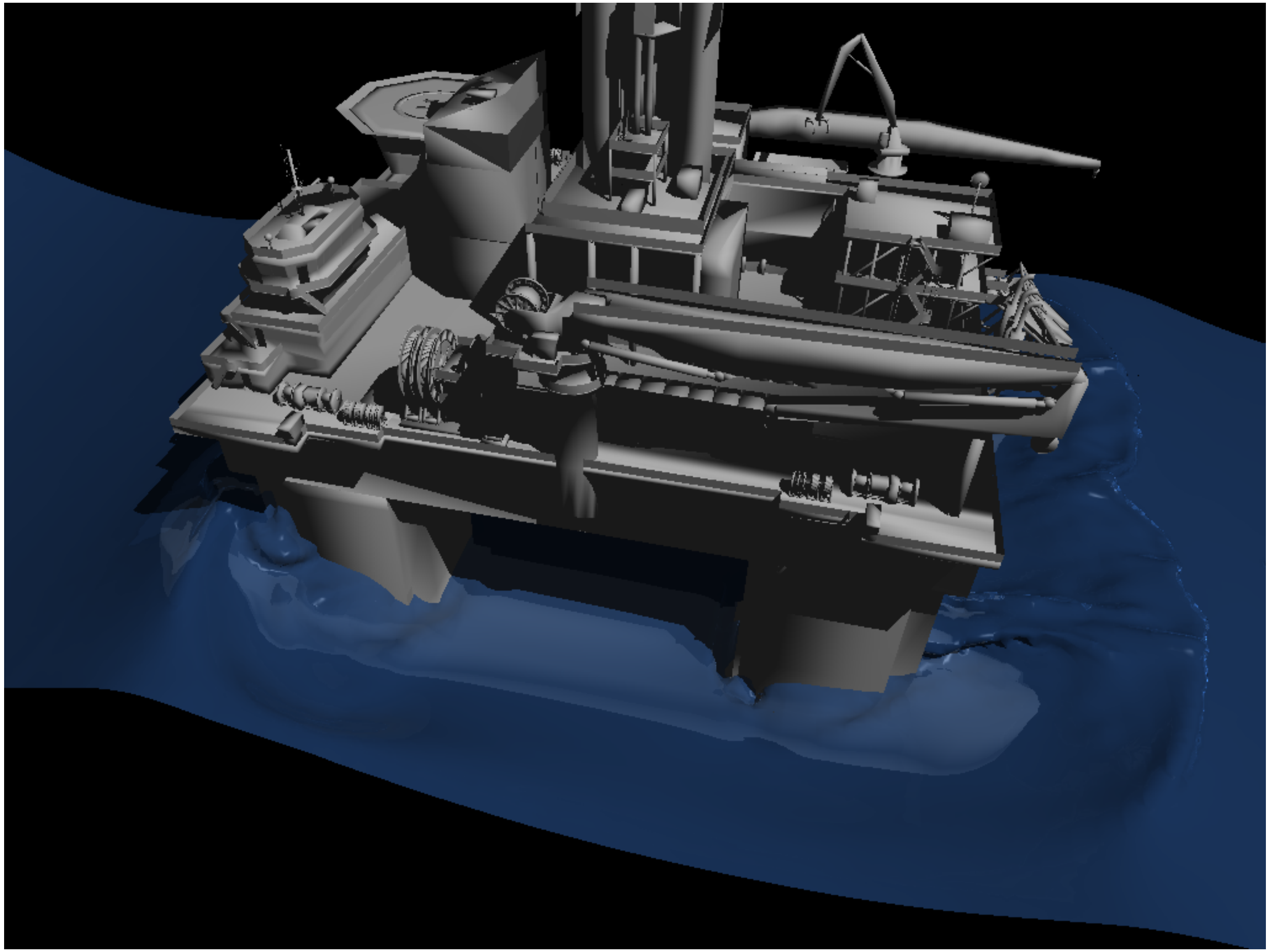
Free surface flow passing offshore oil platform



A field of normal stresses projection at x-direction

K.Nikitin et al. *An adaptive numerical method for free surface flows passing rigidly mounted obstacles.* Computers & Fluids, 2017, V.148, 56-68

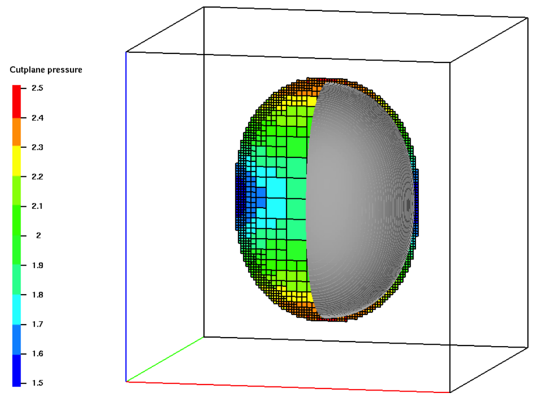
Free surface flow passing offshore oil platform



K.Nikitin et al. *An adaptive numerical method for free surface flows passing rigidly mounted obstacles.* Computers & Fluids, 2017, V.148, 56-68

Freely oscillating droplet problem

Computations for <u>Newtonian</u> fluid



Initial shape:

$$r = r_0(1 + \tilde{\varepsilon} S_2(\frac{\pi}{2} - \theta)),$$

S_2: second spherical harmonic, $r_0 = 1$,
Surface tension: $\varsigma = 1$, $\tilde{\varepsilon} = 0.3$, $K = 1/150$.

Energy balance for Newtonian fluid:

$$\frac{1}{2}\int_{\Omega(t)} |\mathbf{u}(t)|^2 \mathrm{d}\mathbf{x} + K\int_0^t \int_{\Omega(t)} |\mathbf{D}\mathbf{u}|^2 \,\mathrm{d}\mathbf{x}\,\mathrm{d}t' + |\Gamma(t)| = \frac{1}{2}\int_{\Omega(t)} |\mathbf{u}(0)|^2 \mathrm{d}\mathbf{x} + |\Gamma(0)|,$$

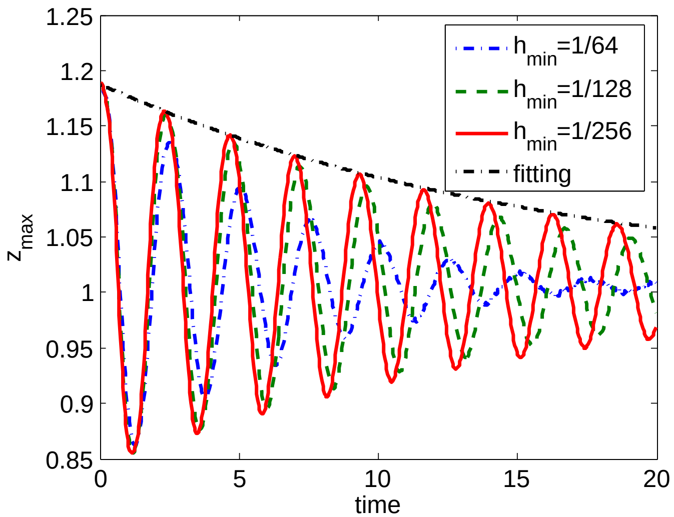
here $|\Gamma(t)| = meas(\Gamma(t))$.

For the Newtonian case:

Top tip trajectories on z axes

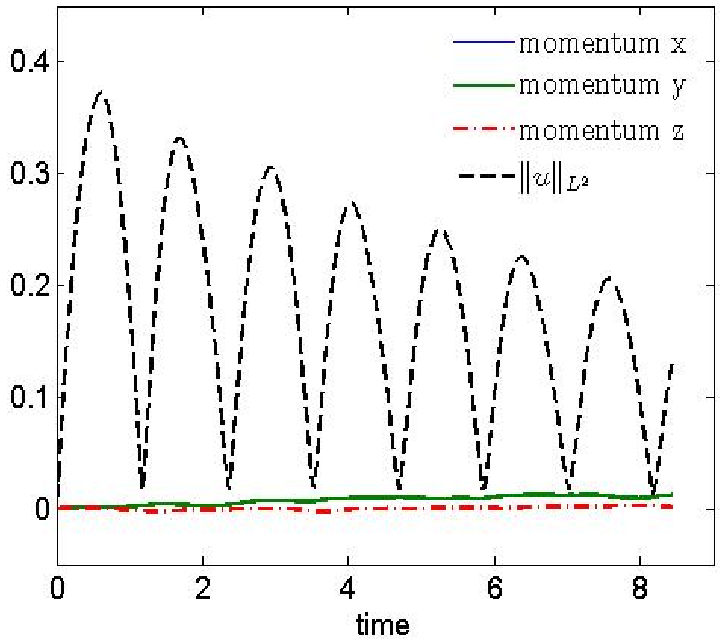
and

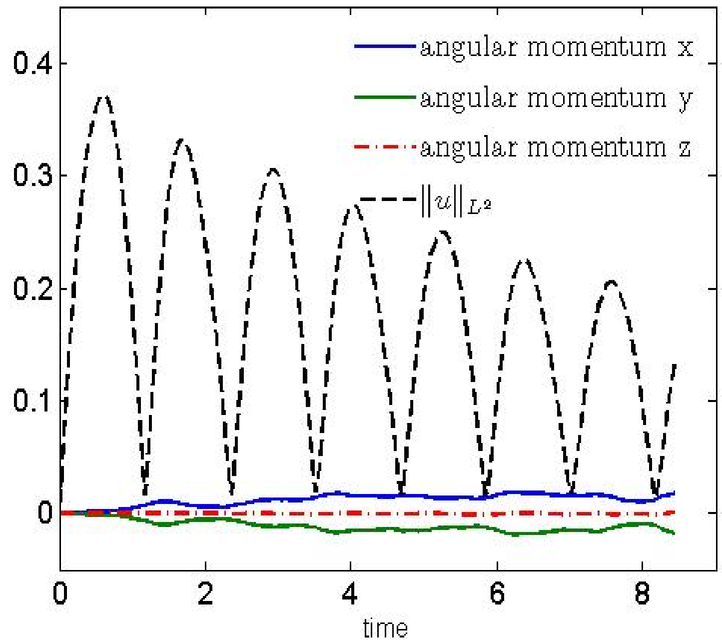
fitting curve $z = r_\infty + c\exp(-\frac{t}{\delta})$
with $\delta = 16.2 \;\Rightarrow\;$ numerical dissipation is an issue.



Computations for Newtonian fluid

Momentum and angular momentum of freely oscillating droplet





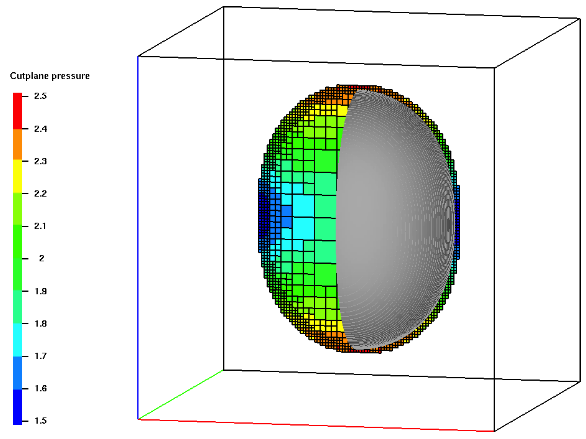
Freely oscillating droplet problem

Computations for <u>*Herschel-Bulkley*</u> *fluid,*
$n = 1 \Rightarrow$ *Bingham*

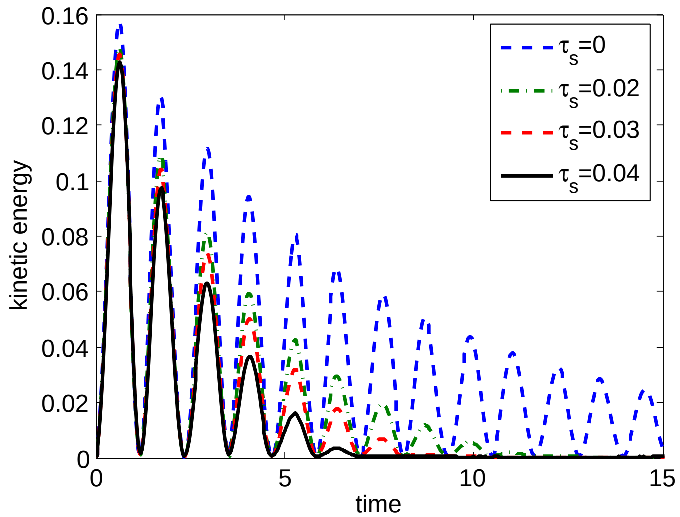
Viscoplastic case, $\tau_s > 0$

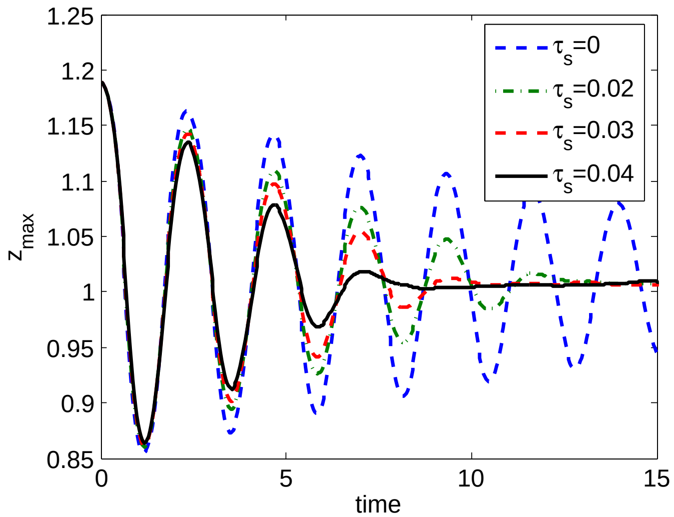
$\Downarrow$

Finite cessation times?



The kinetic energy decay (left) and top tip trajectories (right) for different stress yield parameter values, $\tau_s \in \{0,\ 0.02,\ 0.03,\ 0.04\}$.





Numerical analysis challenge:

For the explicit time stepping treatment of visco-plastic term $\mathbf{div}\mu_\varepsilon\mathbf{D}\mathbf{u}$ one might expect stability condition:

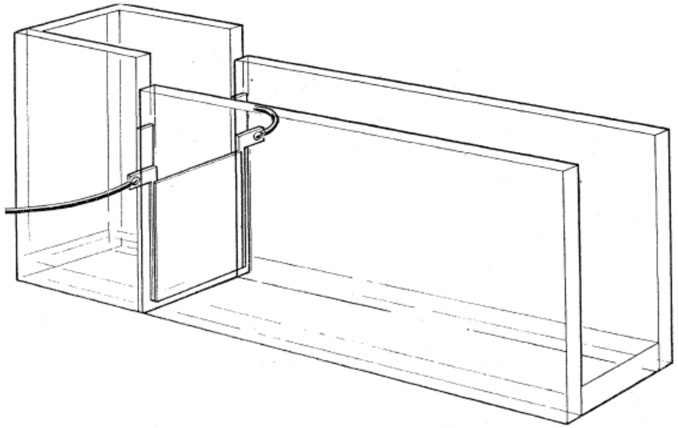
$$\Delta t \le \frac{h^2_{\mathsf{min}}}{\mathsf{max}\,|\mu_\varepsilon|}, \qquad (\text{in practice } \mathsf{max}\,|\mu_\varepsilon| \gtrsim 10^7).$$

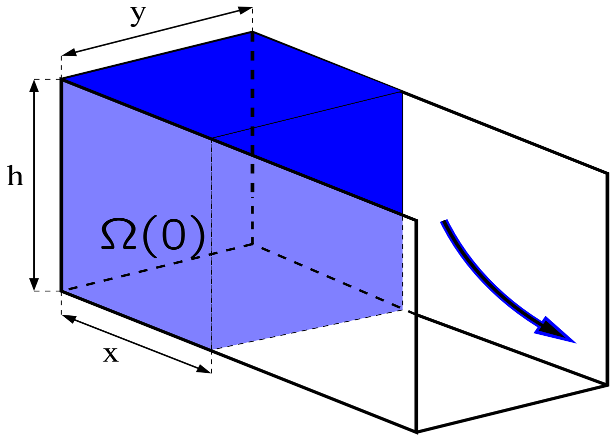
Was not observed in practice!

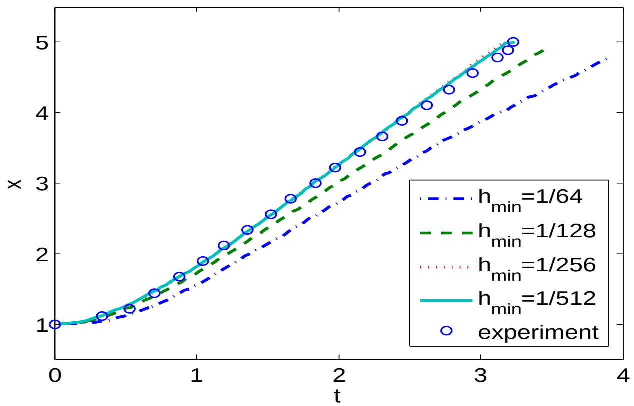
Observed stability can be related with the non-linear dependence of μ_ε on $\mathbf{u}$ (for large μ_ε the solution is constrained)...
More rigorous explanation would be very desirable.

Computations for <u>Newtonian</u> fluid

Comparison to experiment: Collapsing water column (a.k.a. broken dam).







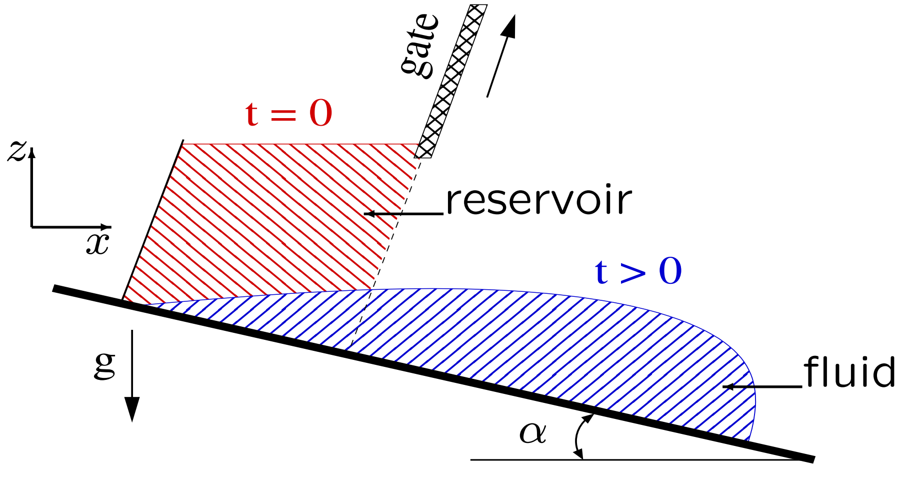
⟸ Convergence to experimental data for refined adapted meshes

Calculation for NSE + level set on octree dynamic meshes

versus

J. Martin and W. Moyce, *An experimental study of the collapse of liquid columns on a rigid horizontal plane*, Philos.Trans.R.Soc.Lond.Ser.A, 244 (1952).

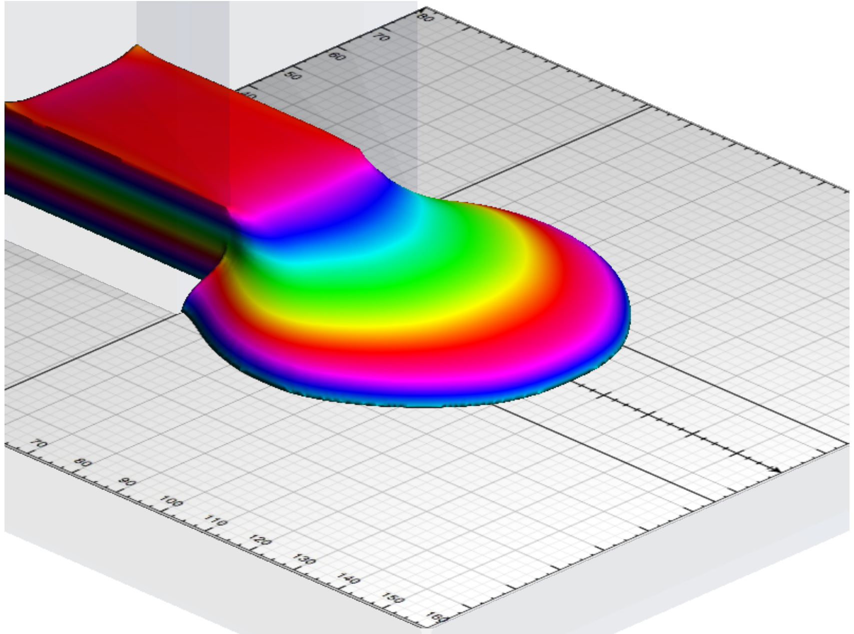
Computations for Herschel-Bulkley fluid

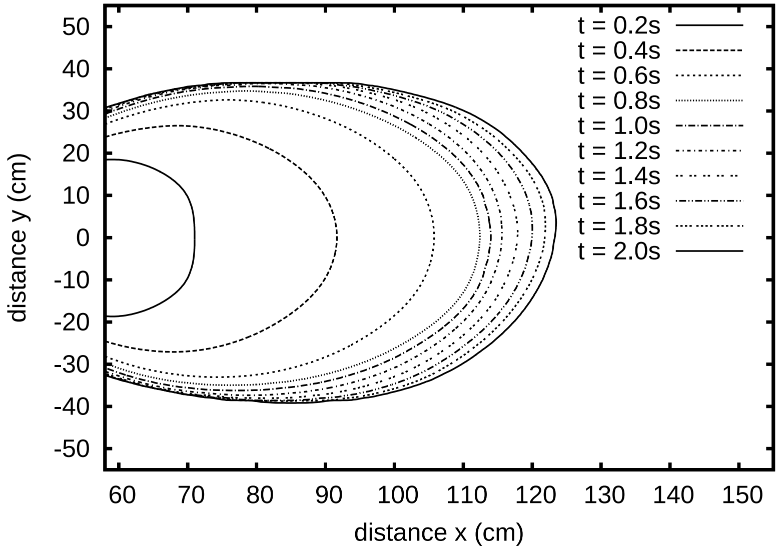


The sketch of the flow configuration: viscoplastic fluid flows over incline planes.

Compare to experimental results with Carbopol Ultrez 10 gel from S. Cochard, C. Ancey, Experimental investigation of the spreading of viscoplastic fluids on inclined planes, J. Non-Newtonian Fluid Mech. 158 (2009).

Computations for Herschel-Bulkley fluid



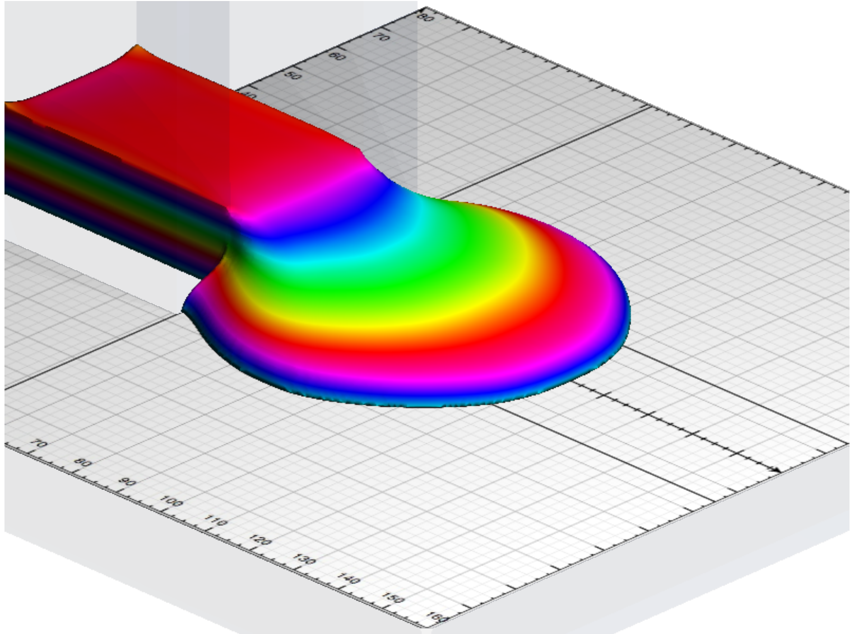


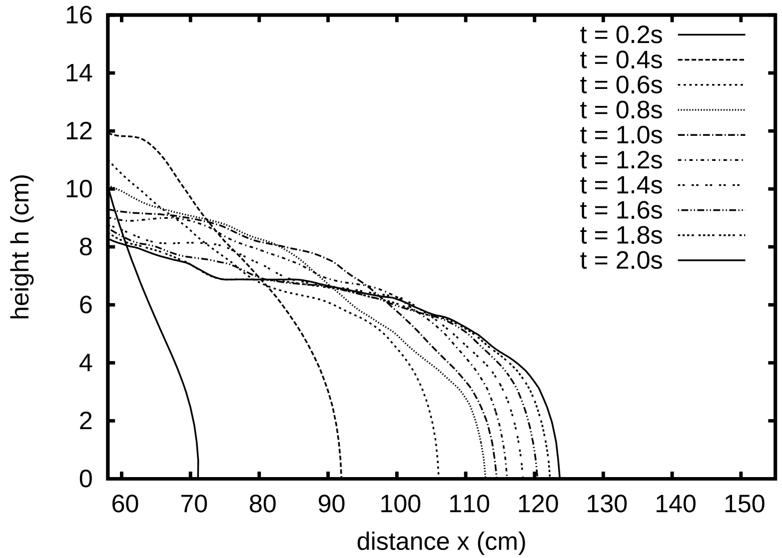
Flow animation: viscoplastic fluid flows over incline planes.
Plots: evolution of the contact line of the free-surface
Parameters: $\alpha = 12^o$, $K = 47.68 Pas^{-n}$, $n = 0.415$, $\tau_s = 89 Pa$.

Citation from Cochard & Ancey "... we observed two regimes: at the very beginning ($t < 1s$), the flow was in an inertial regime; the front velocity was nearly constant. Then, quite abruptly, a pseudo-equilibrium regime occurred, for which the front velocity decayed as a power-law function of time."

S. Cochard, C. Ancey, Experimental investigation of the spreading of viscoplastic fluids on inclined planes, J. Non-Newtonian Fluid Mech. 158 (2009).

Computations for Herschel-Bulkley fluid



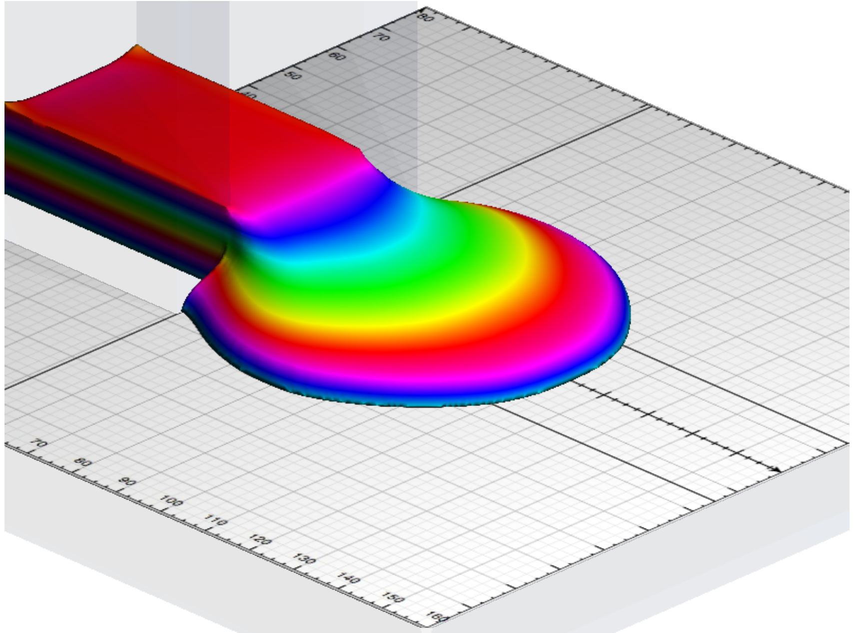


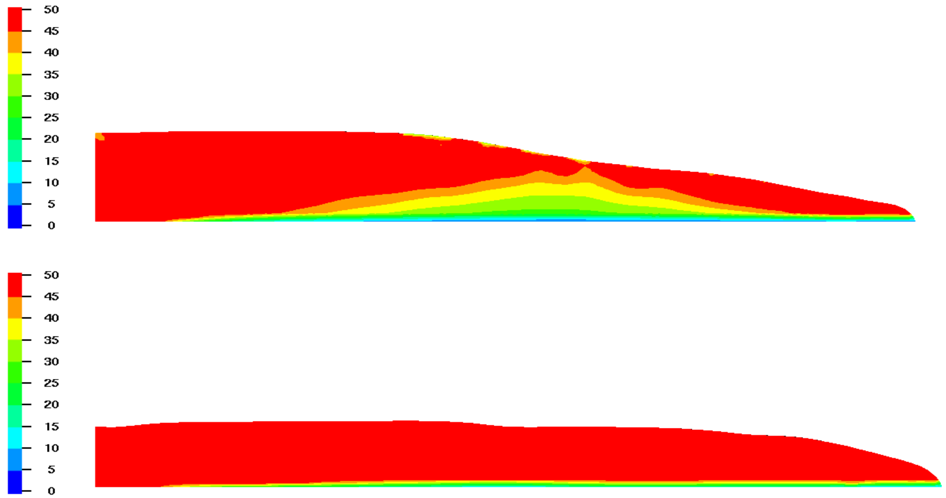
Flow animation: viscoplastic fluid flows over incline planes.
Plots: evolution of the midplane flow-depth profile
Parameters: $\alpha = 12^o$, $K = 47.68 Pas^{-n}$, $n = 0.415$, $\tau_s = 89 Pa$.

Citation from Cochard & Ancey "... we observed two regimes: at the very beginning ($t < 1s$), the flow was in an inertial regime; the front velocity was nearly constant. Then, quite abruptly, a pseudo-equilibrium regime occurred, for which the front velocity decayed as a power-law function of time."

S. Cochard, C. Ancey, Experimental investigation of the spreading of viscoplastic fluids on inclined planes, J. Non-Newtonian Fluid Mech. 158 (2009).

Computations for Herschel-Bulkley fluid





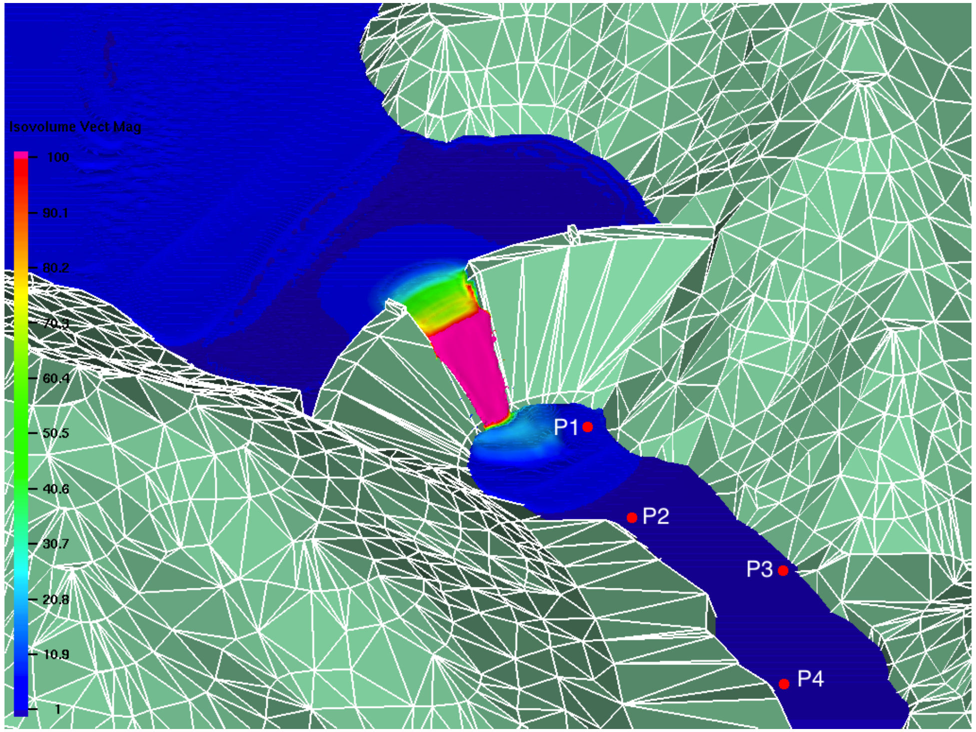
Flow animation: viscoplastic fluid flows over incline planes.
Plots: Effective viscosity μ_ε on midplane at $t = 0.6s$ and $t = 1s$
Parameters: $\alpha = 12^o$, $K = 47.68 Pas^{-n}$, $n = 0.415$, $\tau_s = 89 Pa$.

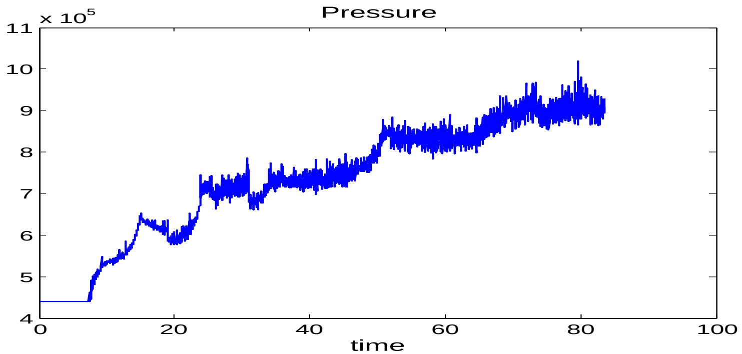
The existing shallow-layer theory distinguishes *yielding region close to the bottom boundary* and the pseudo-plug region,where the fluid is considered solid up to higher order terms with respect to the layer thickness.

N. J. Balmforth et al., Viscoplastic flow over an inclined surface, J. Non-Newtonian Fluid Mech. 139 (2006)

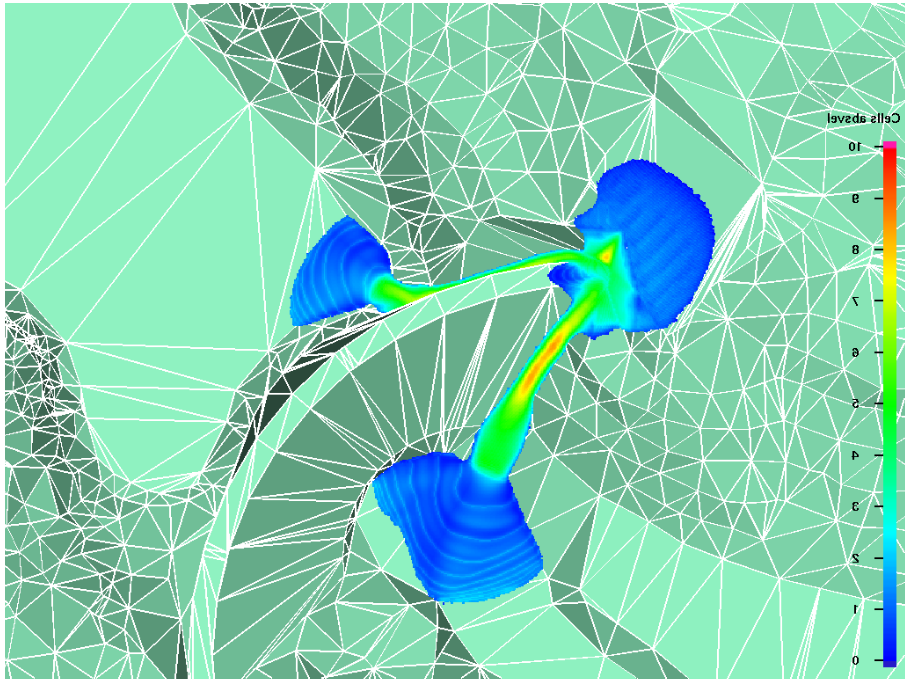
Newtonian fluid

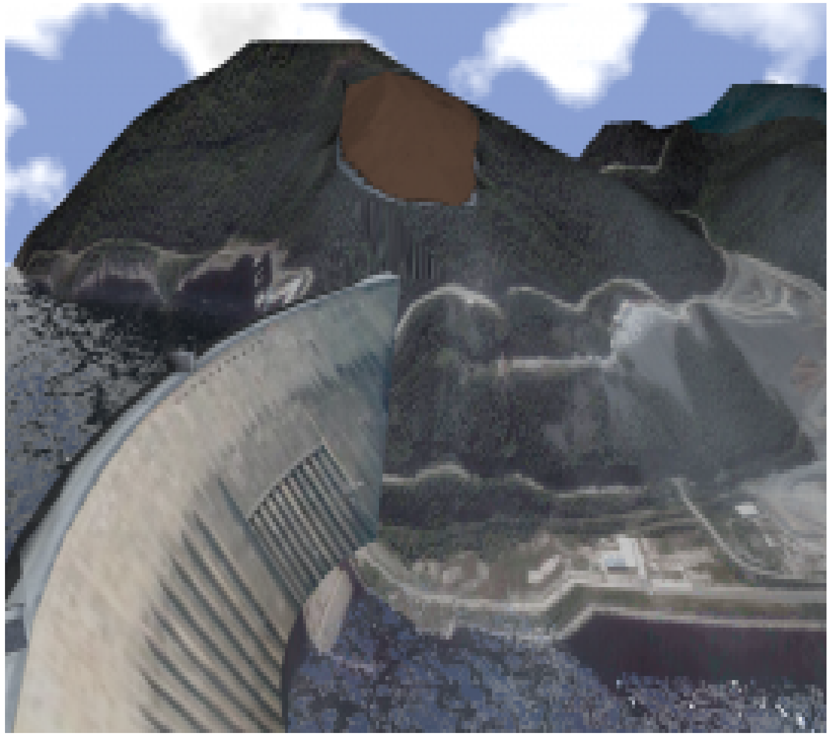
Sayano-Shushenskaya Dam Break
(real-life topography)

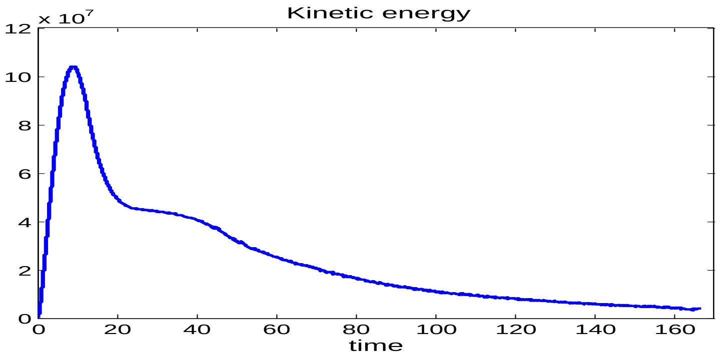




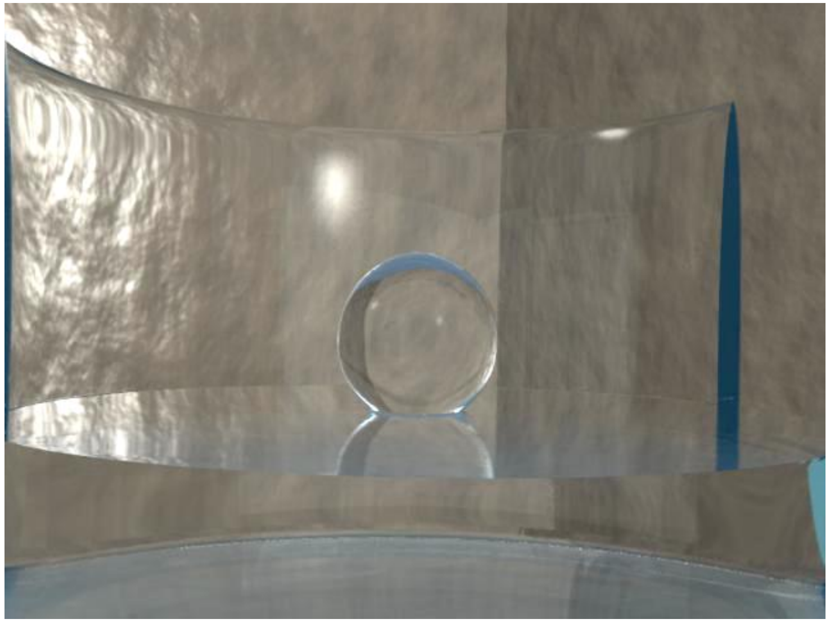
Sayano-Shushenskaya Dam Landslide
(real-life topography)

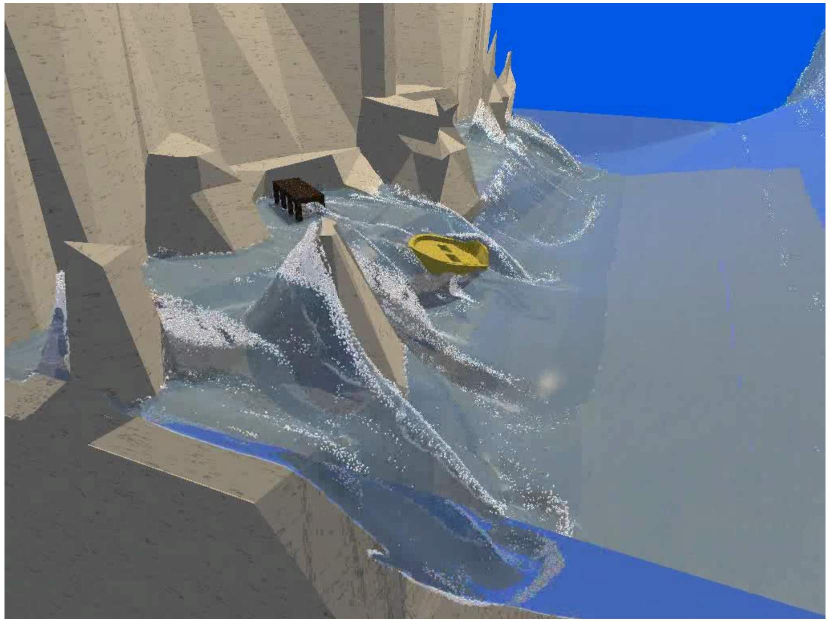


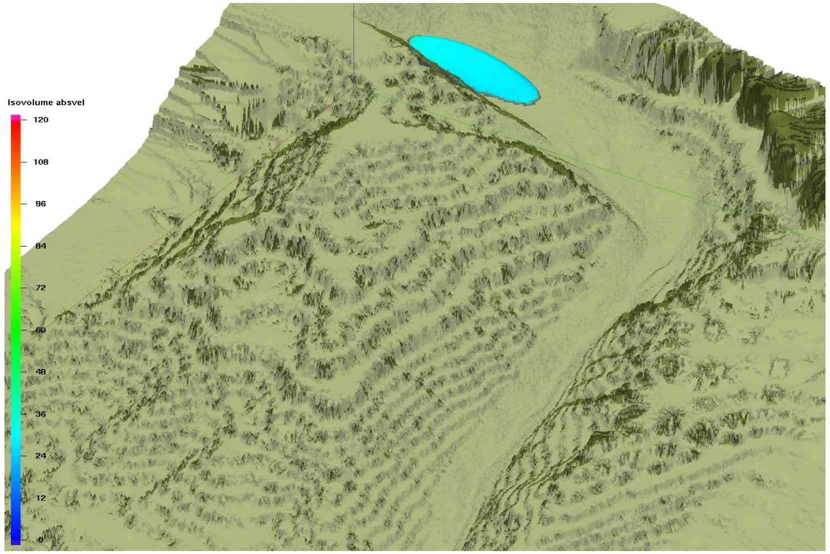


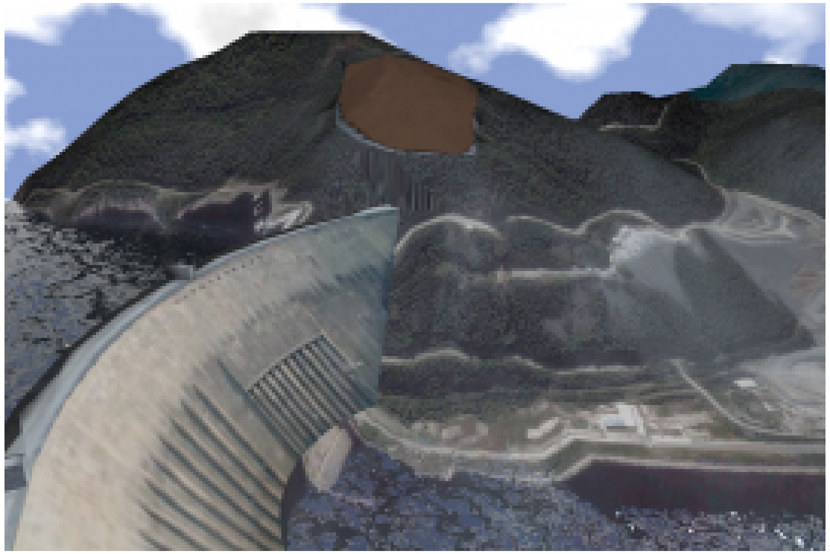


Newtonian and non-Newtonian fluids









Much more (papers, flows animations) on:

www.inm.ras.ru/research/freesurface

Acknowledge support of RFBR, RNF and MSE