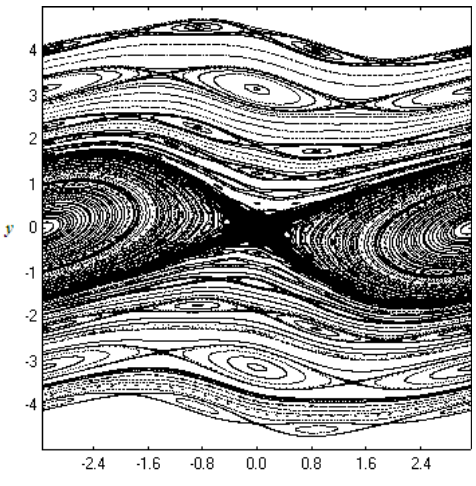
«Three forms of dynamical chaos»

S.Gonchenko (Lobachesky Univ., Nizhny Novgorod, Russia)

Нелинейные Волны 2020, 4 марта

3 Forms of Dynamics

I) Conservative (Hamiltonian)

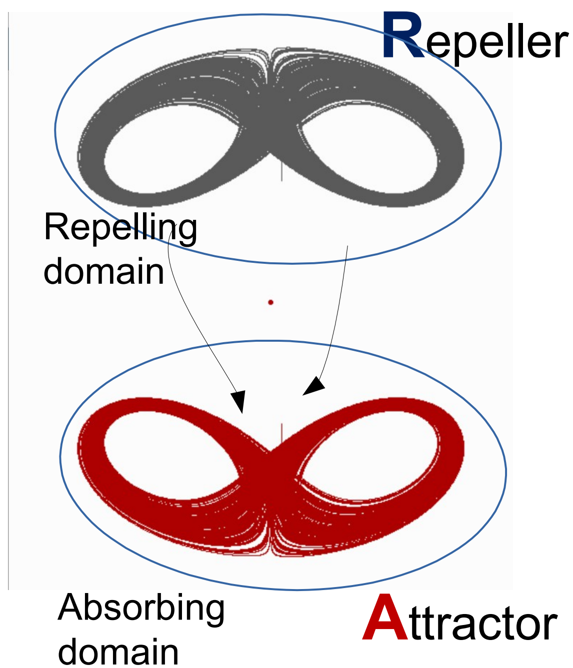


$$\begin{cases} x_{n+1} = x_n + y_{n+1} \\ y_{n+1} = y_n + k \sin x_n \end{cases}$$

$$\boxed{A = R}$$

Chaos “spread” over the phase space

II) Dissipative

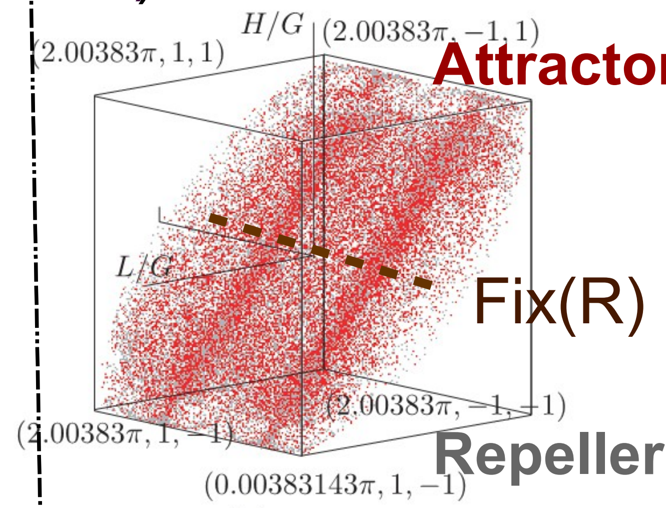


$$\boxed{A \cap R = \varnothing}$$

Chaos “is concentrated” on fractal invariant sets

III) Mixed

New!



$$\boxed{\begin{array}{c} A \cap R = RC \neq \varnothing \\ A \neq R \end{array}}$$

Chaos is both “spread” and “concentrated”

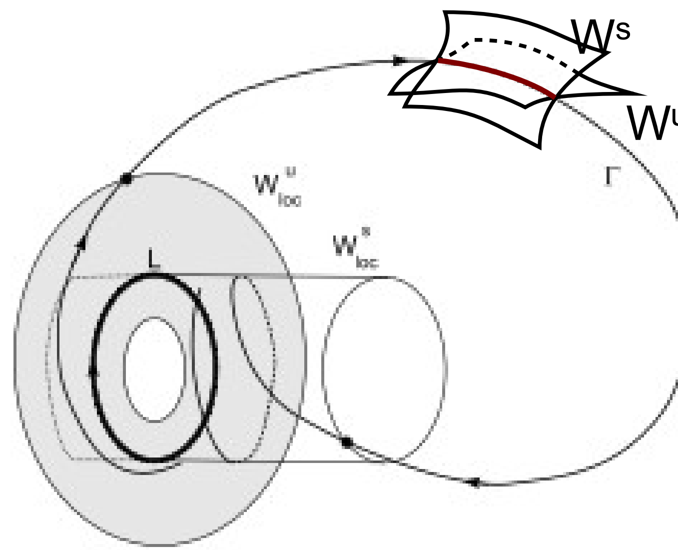
I) Conservative (Hamiltonian) Chaos

Chaos “spread” over the phase space: elliptic islands are totally replaced with non-uniform hyperbolicity

H. Poincare (<u>1893</u>) – the 'oldest' type of dynamical chaos

Poincare homoclinic orbit

Jules Henri Poincare
French mathematician, physicist, astronomer and science theorist
April 29, 1854 - July 17, 1912

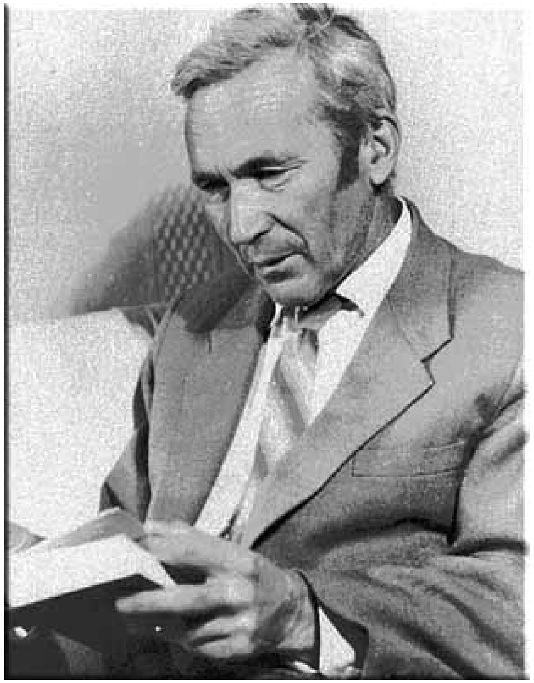


George David Birkhoff
(21.03.1884 – 12.11.1944)

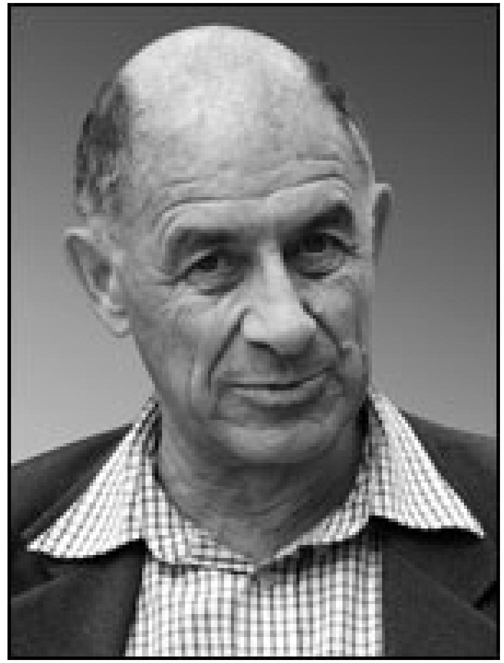
American mathematician, best known for his work on statistical mechanics and ergodic theory.

Jacques Hadamard
(8.12.1865 – 17.10.1963)

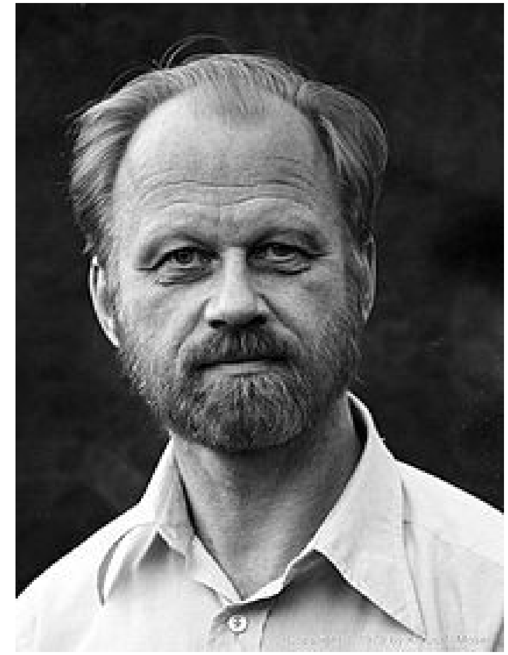
French mathematician and mechanic. The author of many fundamental works in algebra, geometry, functional analysis, differential geometry, mathematical physics, topology, theory probabilities, mechanics, hydrodynamics, etc.



A.N. Kolmogorov
(25.04.1903 – 20.10.1987)



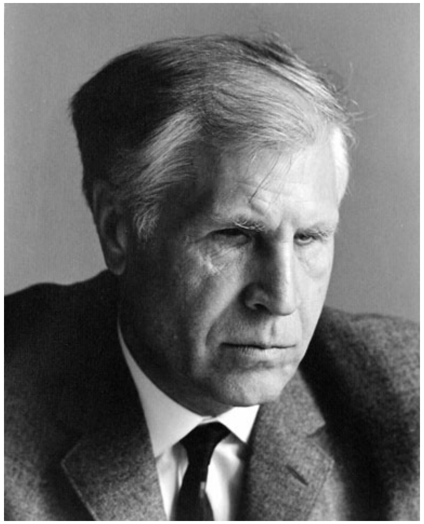
V.I. Arnol'd
(12.06.1937 – 03.06.2010)

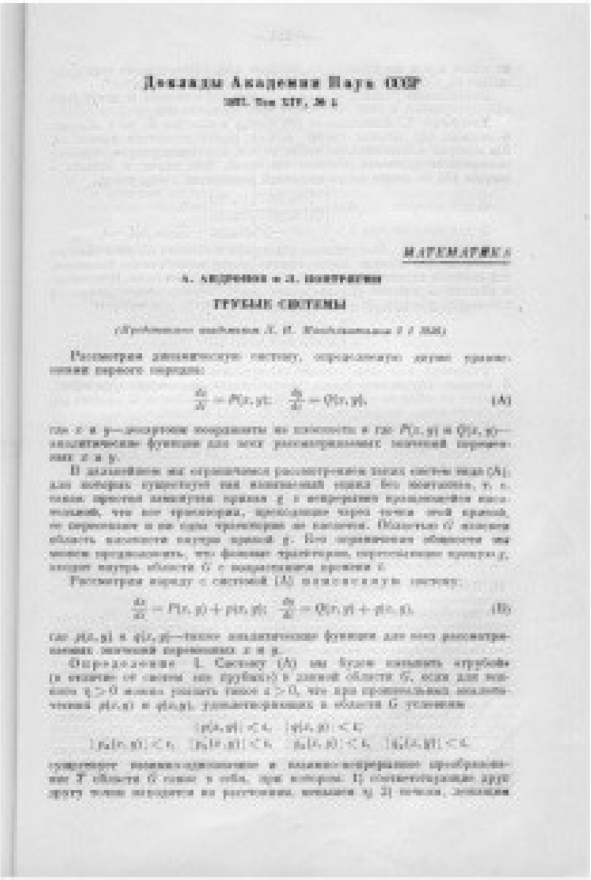


J. Moser
(04.07.1928 – 17.12.1999)

The Kolmogorov–Arnold–Moser theory, or KAM theory — named after its creators, A. N. Kolmogorov, V. I. Arnold, and J. Moser, relates to a branch of the theory of dynamical systems that studies small perturbations of almost periodic dynamics in Hamiltonian systems and related topics - in particular, in dynamics of symplectic mappings. Its main theorem, the Kolmogorov–Arnold–Moser theorem, or KAM-theorem says about the preservation of majority of invariant tori in phase space when small perturbations of integrable Hamiltonian systems.

II) Dissipative dynamics





Доклады Академии Наук СССР

МАТЕМАТИКА

А. АНДРОНОВ и Л. ПОНТРЯГИН

ГРУБЫЕ СИСТЕМЫ

$$\frac{dx}{dt} = P(x, y); \quad \frac{dy}{dt} = Q(x, y). \qquad (A)$$

Alexandr Alexandrovich **Andronov** (1901—1952)

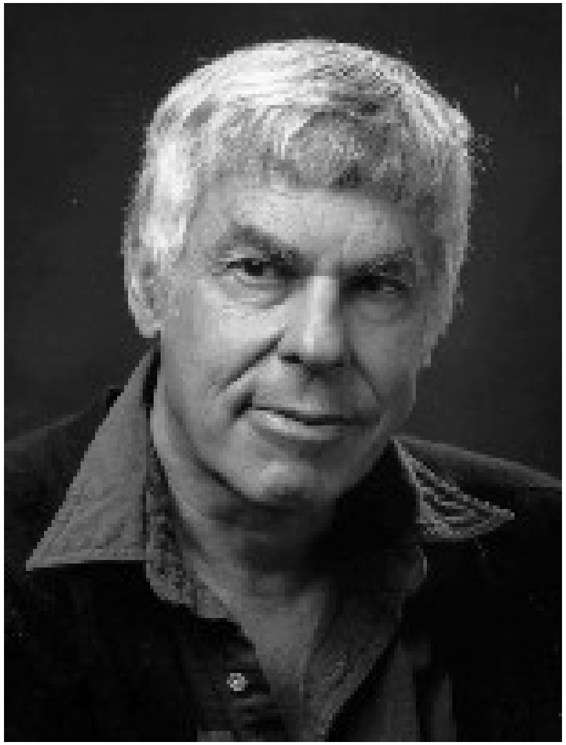
Soviet physicist, mechanic
and mathematician,
Academician of the USSR
Academy of Sciences.
Founder of the theory of

Lev Semenovich **Pontryagin** (1908-1988)

Soviet mathematician,
one of the greatest
mathematicians of the
20th century,
Academician
of the Academy of
Sciences

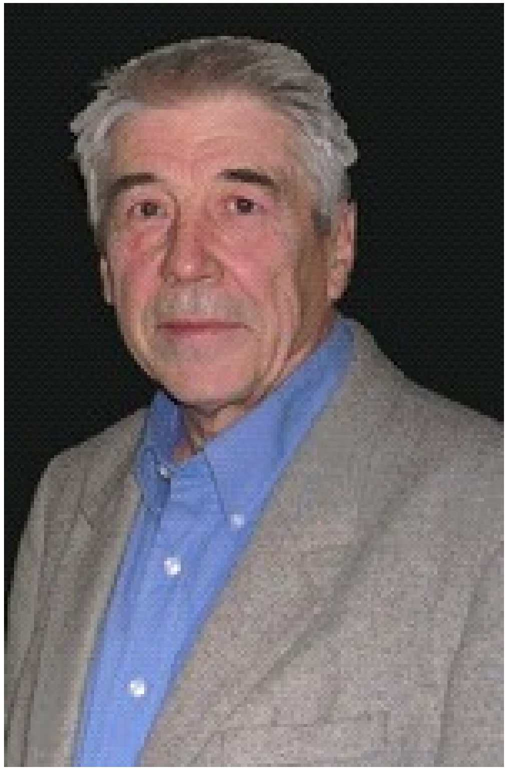
Poincare-Birkhoff problem:

to give a description of orbits entirely lying in a neighbourhood of a transverse Poincare homoclinic orbit



S. Smale
(born 15.07.1930)

D.V. Anosov
(30.11.1936 – 05.08.2014)



L.P. Shilnikov
(17.12.1934 – 26.12.2011)

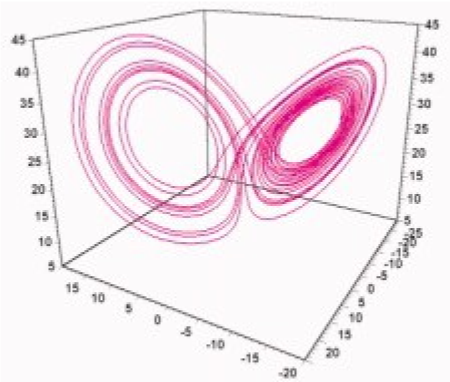
II) Strange attractors

<u>Genuine SA:</u> all orbits have a positive maximal Lyapunov exponent

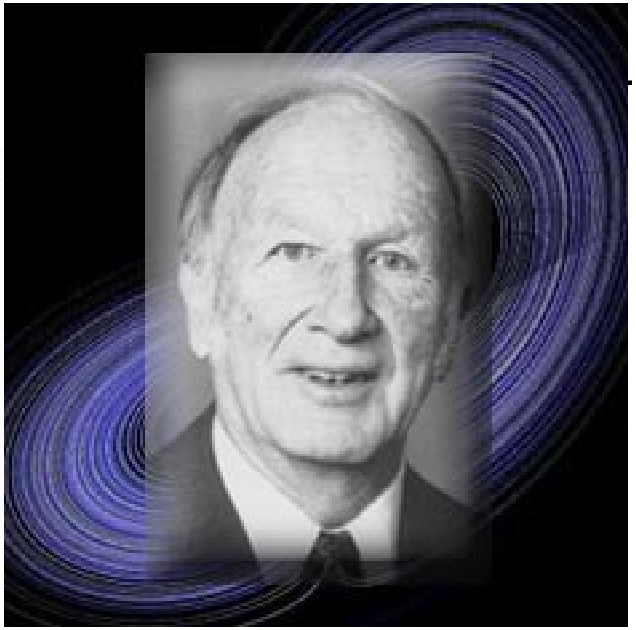
ex.: hyperbolic SA, the Lorenz SA, pseudohyperbolic SA <u>(Lecture 2)</u>

<u>Quasiattractors:</u> some orbits in *D* have maximal L-exp equal 0
(coexisting SA-behavior with stable reqular one (e.g. per. sinks) windows of stability in the parameter space.

E.Lorenz (<u>1963</u>); …

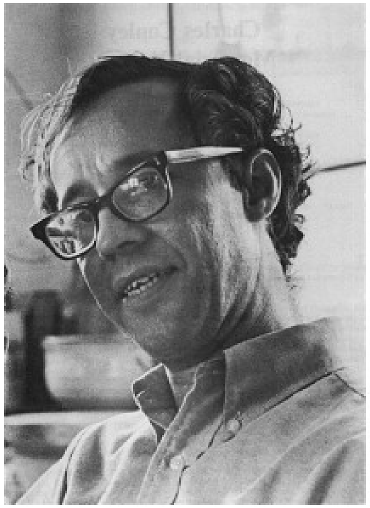


Lorenz attractor



Edward Norton Lorenz –
(23.05.1917 – 16.04.2008)

American mathematician and meteorologist, one of the founders
of chaos theory, the author of the expression "butterfly effect", as well as the creator of the Lorenz attractor



C.K. Conley
(26.09.1933 – 20.11.1984)

American mathematician. He made a significant contribution to the theory of DS (Conley index, Conley's theorem – “the main theorem of DS”, etc.)

D. Ruelle
(**b**orn: 20.08.1935)

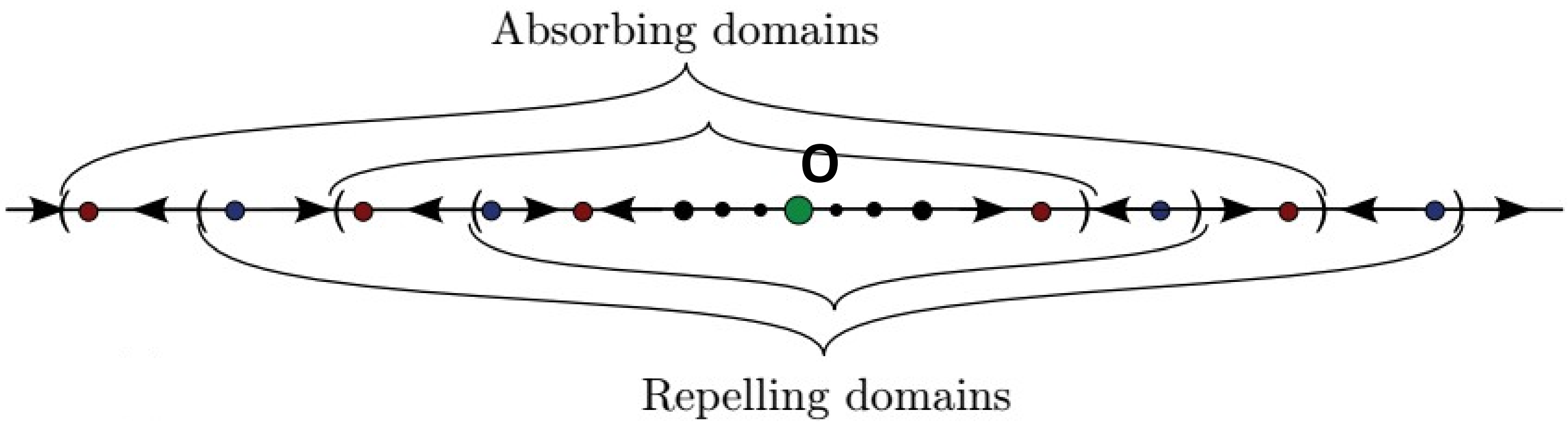
Belgian mathematician and physicist working in areas of statistical physics and theory of dynamics systems.

F. Takens
(1940 - 2010)

Dutch mathematician. He made a significant contribution to the theory of dynamical systems, chaos theory, fluid mechanics

Why can an **A**ttractor intersect with a **R**epeller if (by definition) they can never intersect?

Simple example:



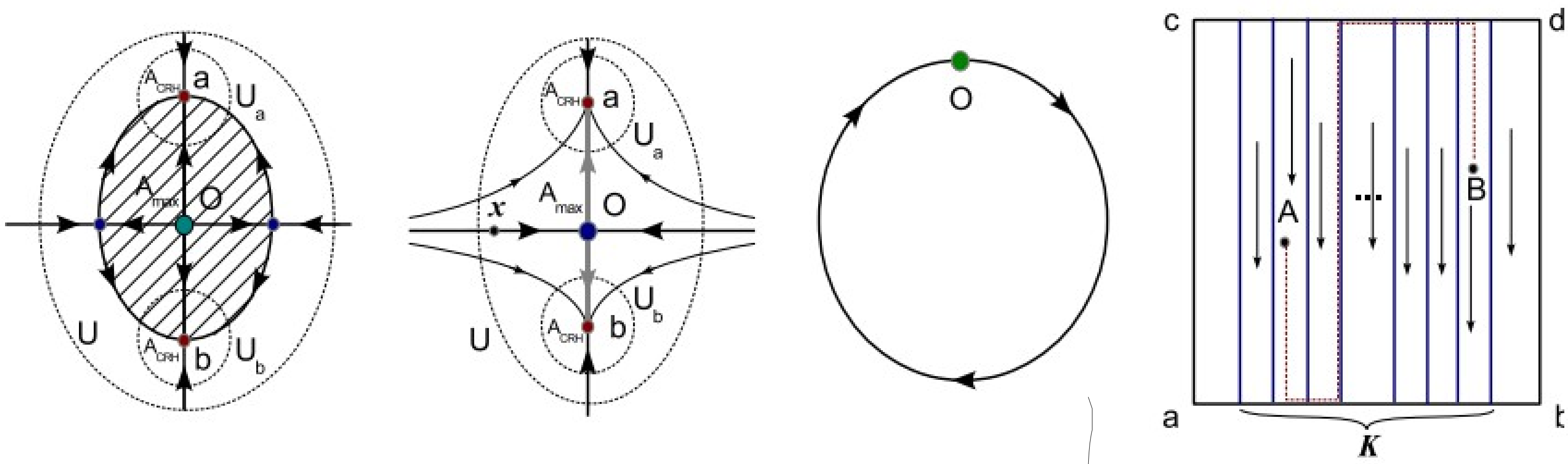
Main question: What is attractor (repeller) ?

We (all) want the attractor to be a closed and stable invariant set

There are many different definitions of attractor. For example.

Maximal attractor: $$A_{max} = \bigcap_{n=0}^{\infty} T^n(D)$$

Milnor attractor: **a closed invariant and minimal by embedding set in D that contains w-limit points of forward orbits of almost all (Lebesque mes 1) of points from D.**



(a) (b) (c) (d)

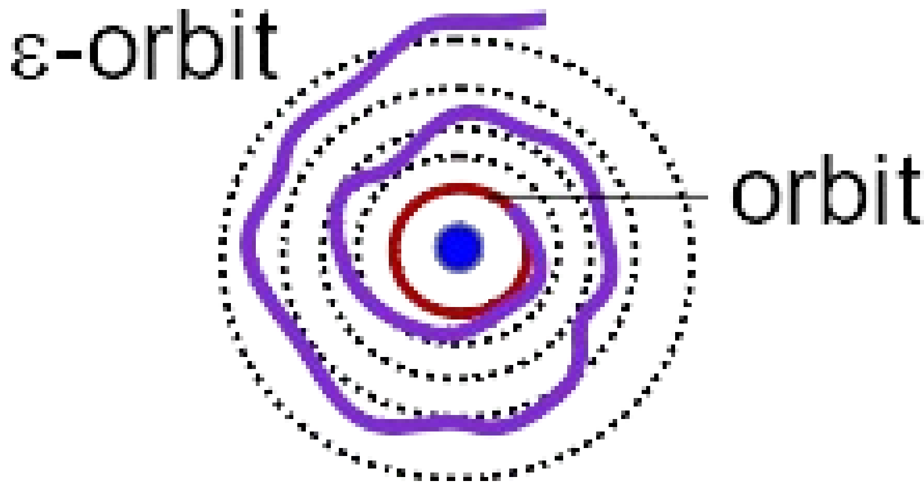
Not attractors

AM is not topological invariant

Definition 1. Let f : M → M be a map $y_{i+1} = f(y_i)$.

A sequence $\{x_n\}$, , $n \in Z$, is call **orbit** , if $x_{n+1} = f(x_n)$,

ε-orbit , if $\boxed{\rho(x_{n+1}, f(x_n)) < \varepsilon}$.



Definition 2. We will call a point *y* ***attainable*** from a point *x* if for any ε > 0

there exist an ε-orbit which starts at *x* and ends at *y*.

Definition 3. A closed invariant set B is called ***chain-transitive***, if every point

of B is attainable from any other point of B.

Definition 4. A closed invariant set B is called ***ε-stable*** if for every open neighborhood *U(B)* there exists a neighborhood *V(B)* such that, for all sufficiently small ε>0, the ε-orbits which start in *V(B)* never leave *U(B)*.

For any definition of attractor, we should consider it as

a **closed**, **stable** and **invariant** set !

Stability – under permanently acting perturbations (total stability or ***ε-stability***)

CRH-attractor (Conley-Ruelle-Hurley attractor) is a chain-transitive, stable (in fact, ε-stable), closed, invariant set

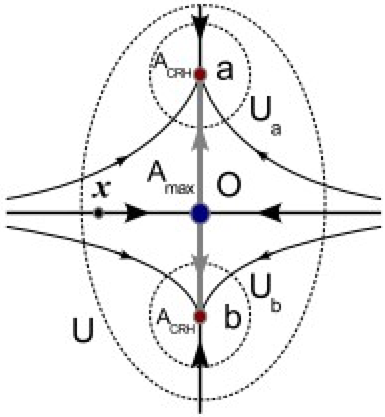
Definition 5. A <u>chain-transitive</u>, <u>stable</u> (in fact, ε-stable), <u>closed</u>, <u>invariant</u> set A_x *(a CRH-attractor)* is called **attractor** of a point *x* if every point of A_x is attainable from *x*.

A set R_x is a repeller of a point *x* if it is an attractor of *x* for the inverse map f^{-1}.

Theorem* ([GT17]).** ***Let x in $A_x \cap R_x$.

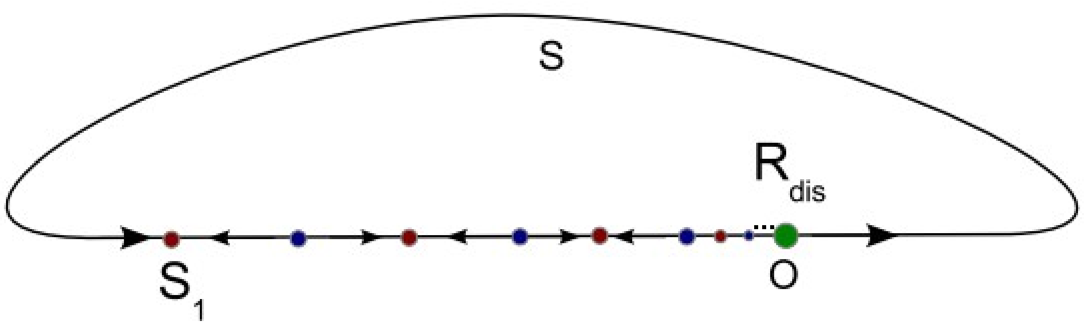
Then $A_x = R_x$.

Note that a point *x* can have several attractors.
If the number # of such attractors is finite, then their union is the **full attractor** of *x*. If # is infinite, the **full attractor** is *the closure of union of all attractors of x*.



This is a closed invariant set *A(x)* but not always stable! Minimal by embedding and stable set containing *A(x)* is **prolongation** of *A(x), i*.e. the set of attainable from *A(x)* points for any arbitrary small ε. We call this set **full Ruelle attractor of *x*.**

Definition 6. The **full attractor** of map *f* is the closure of union of all CRH-attractors of its points, and the **full Ruelle attractor** of *f* is the prolongation of its full attractor.

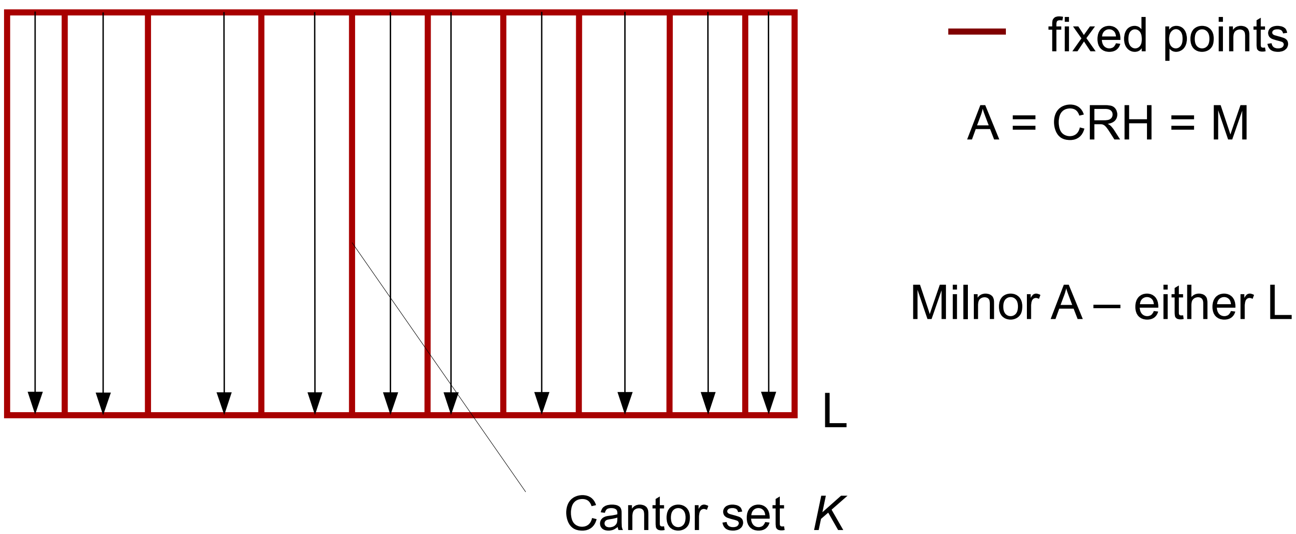


Ex. *FA* is a union of inf. many sinks and *O*.
FRA is a union of *FA* with the arc *S*.

(Topologically) conservative dynamics.

$A_x = R_x = M$ for any point *x*

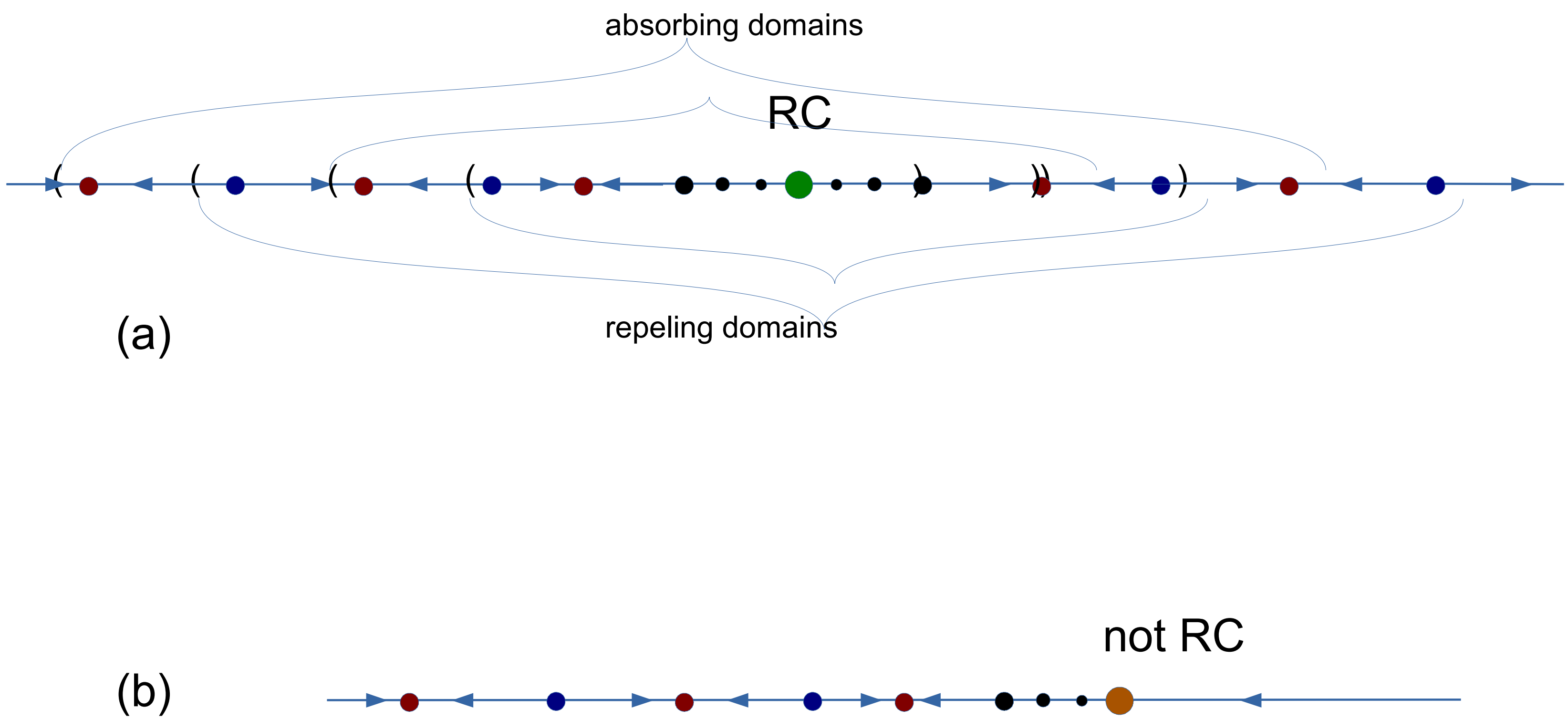
Examples: 1) Hamiltonian systems;
2) Anosov diffeomorphisms
3) A circle map with a saddle-node
4) exotic example (by S.Minkov)



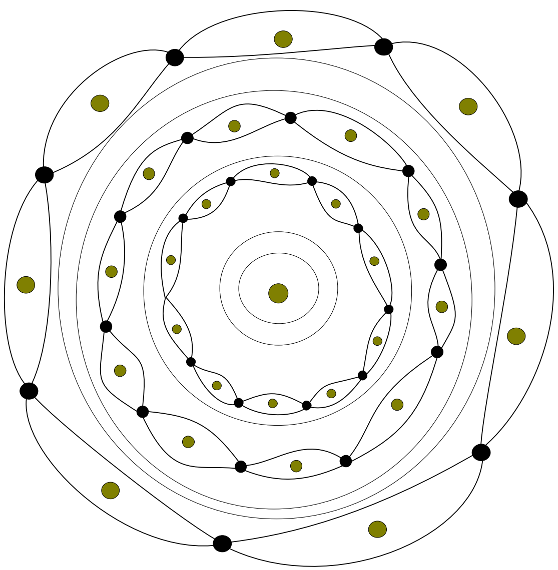
A = CRH = M

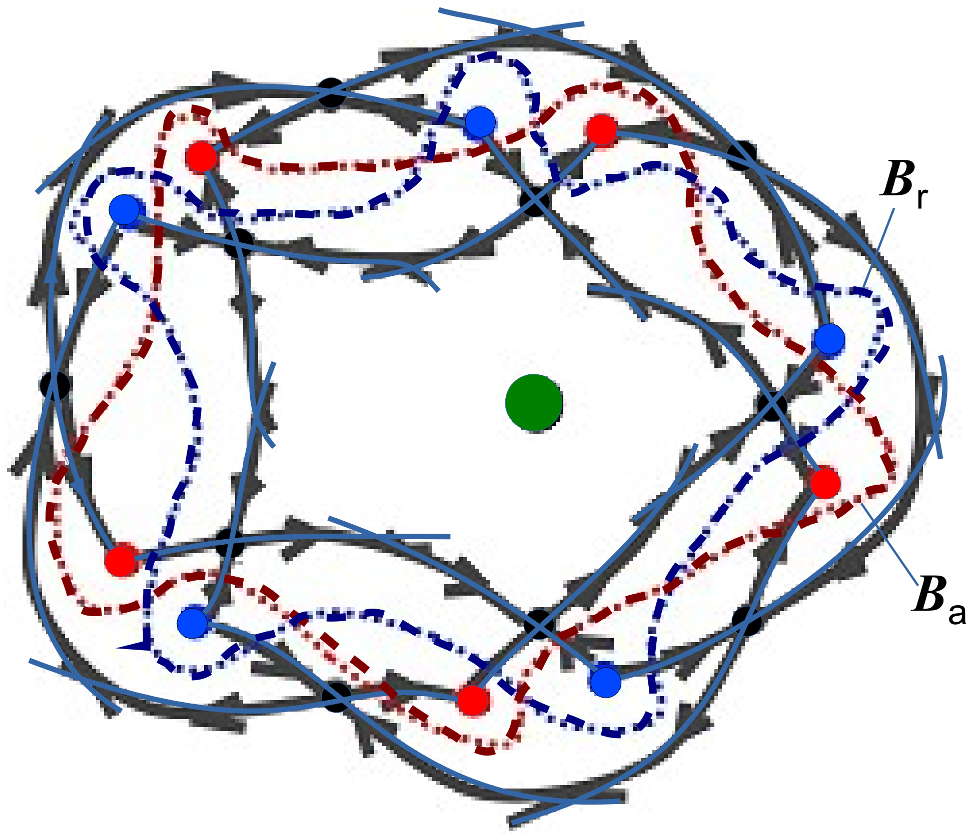
Milnor A – either L or K + L

Mixed dynamics (reversible core).



Mixed dynamics (reversible core).



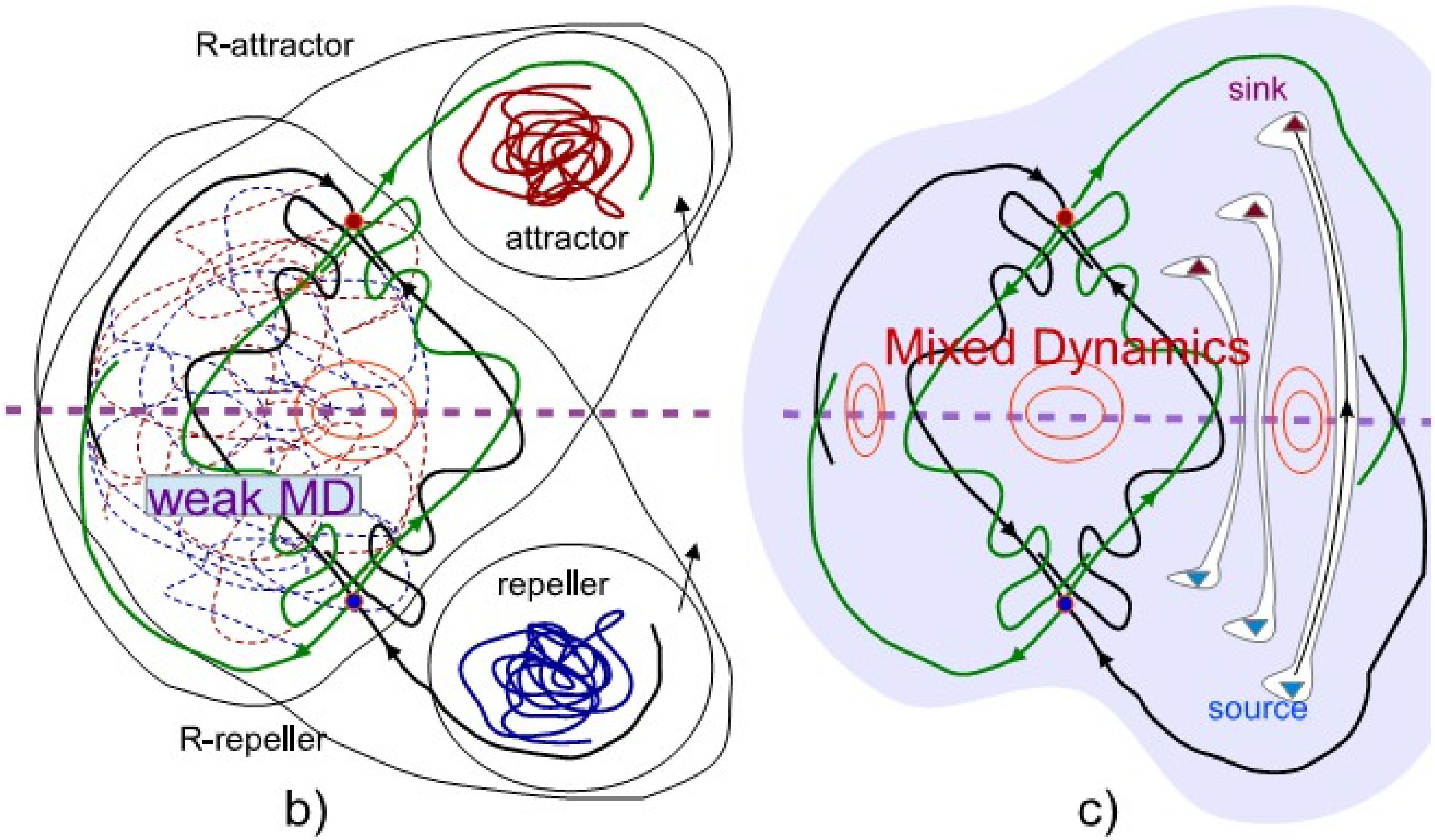


)

Conservative elliptic point

Generic elliptic point for reversible map

RC = ∅ → **Mixed dynamics**



b) phenomenon observed before (“the conservative chaos coexists with the dissipative behavior”) c) a new phenomenon

Attractor and repeller merger

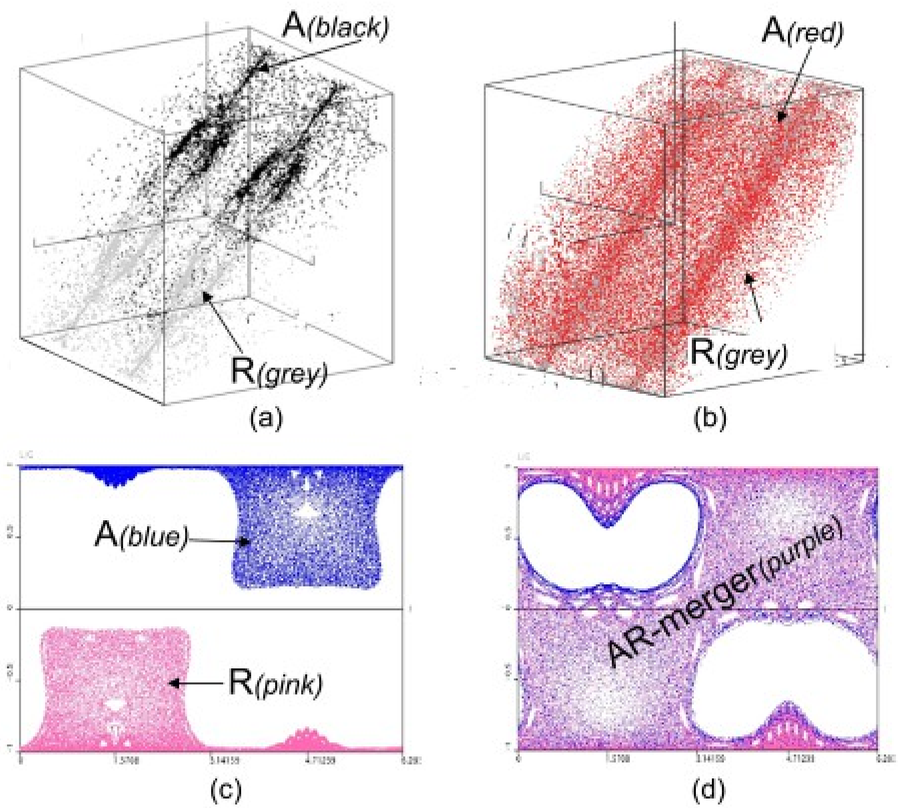


Figure 1: Examples of attractor-repeller merger for the Poincaré map of (a),(b) a model of the Celtic stone, [5], and (c),(d) Chaplygin ball (rubber body), [6]. Here, the numerically obtained attractor (A) and repeller (R) are shown for different values of the energy of the system.

Example of mixed dynamics of dissipative type
(from a paper by A.Kazakov, CHAOS, 2020)

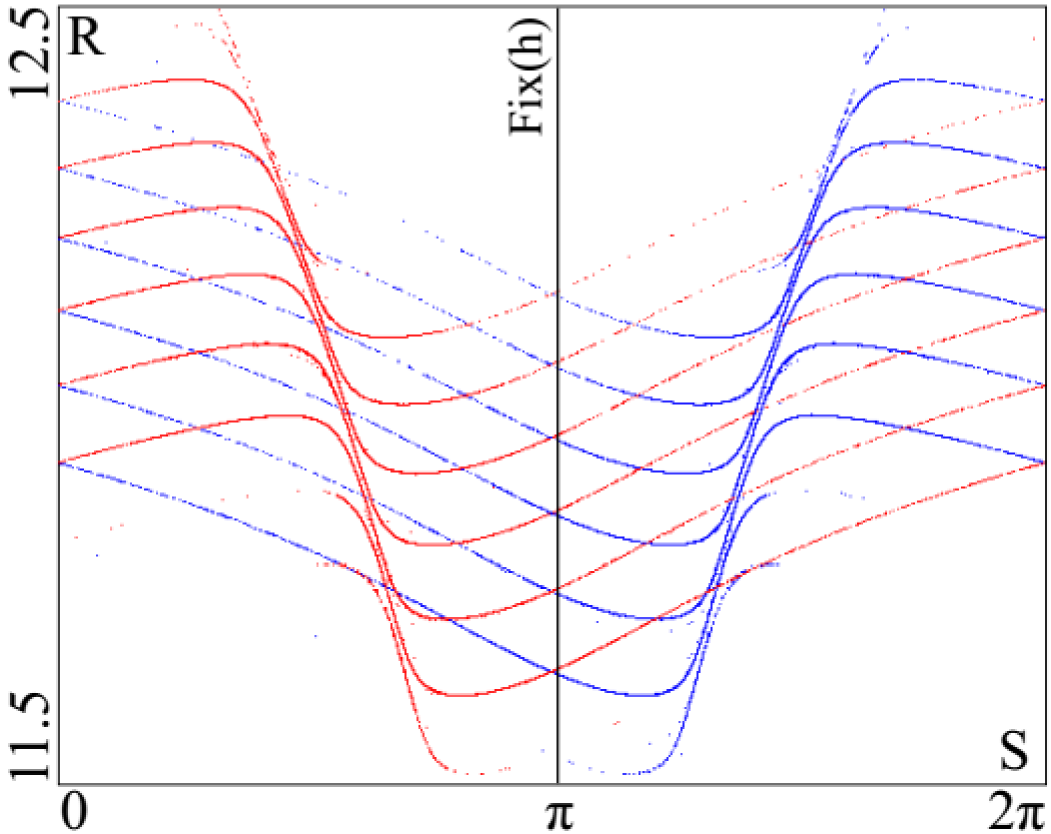


FIG. 7. An illustration for strongly dissipative mixed dynamics after the merger of all eight Hénon-like attractors AH_i and repellers $RH_i, i = 1, \ldots 8$ at . The attractor is presented in blue color, the repeller – in red.

Example 3:
Pikovskii-Topaj model of 4 coupled rotators

In [A.Pikovskii, D.Topaj , 2002] was considered the following model of symmetrically coupled 4 rotators whose frequencies differ on 1

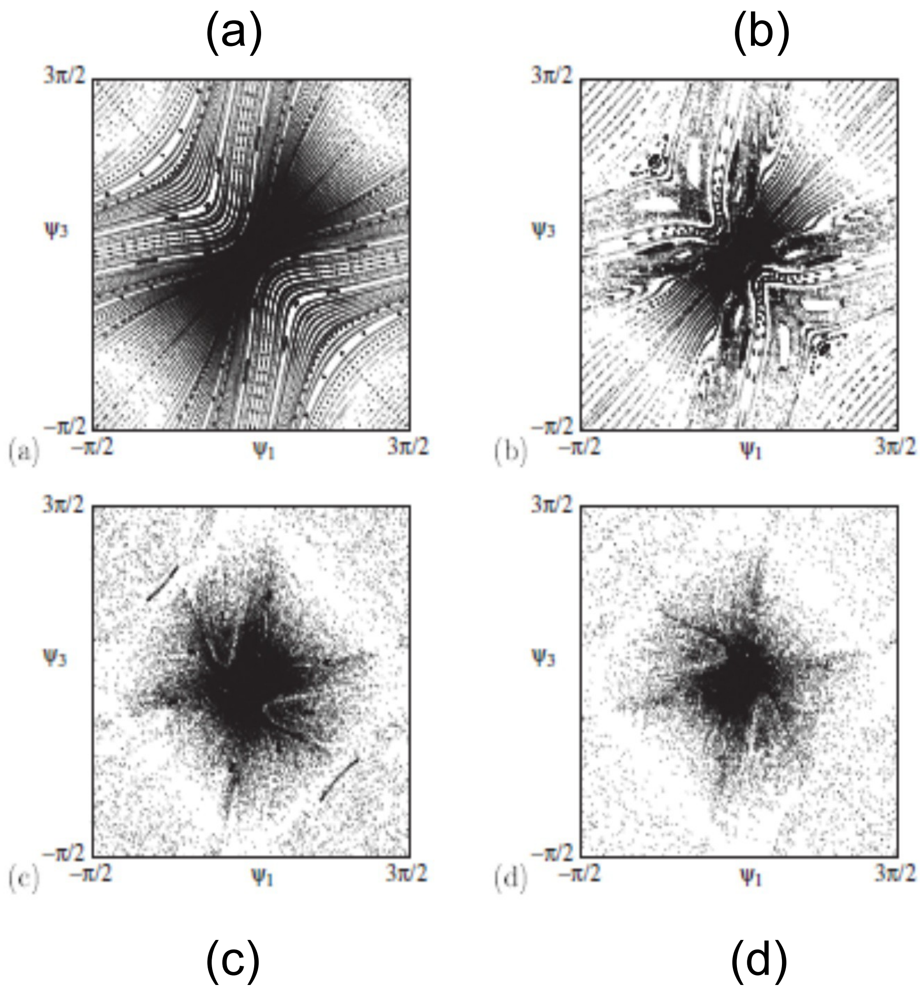
$$\begin{aligned}\dot{\psi}_1 &= 1 - 2\varepsilon\sin\psi_1 + \varepsilon\sin\psi_2 \\ \dot{\psi}_2 &= 1 - 2\varepsilon\sin\psi_2 + \varepsilon\sin\psi_1 + \varepsilon\sin\psi_3 \\ \dot{\psi}_3 &= 1 - 2\varepsilon\sin\psi_3 + \varepsilon\sin\psi_2,\end{aligned} \tag{1}$$

where $\psi_i \in [0, 2\pi), i = 1, 2, 3$, are cyclic variables. Thus, the phase space of (1) is the three-dimensional torus $\mathbb{T}^3$. Note that system (1) is reversible with respect to the involution $\mathcal{R}$:

$$\psi_1 \to \pi - \psi_3 \;\;,\;\; \psi_2 \to \pi - \psi_2 \;\;,\;\; \psi_3 \to \pi - \psi_1, \tag{2}$$

(i.e. equations (1) are invariant under the coordinate change (2) and time reversal $t \to -t$).

Phase portraits of Poincare map $T(\varepsilon)$ for small ε, (a) and (b), and for $T(\varepsilon)$ (c) and $T(\varepsilon)^{-1}$ (d) when ε *is not quite small*



[A.Gonchenko, S.Gonchenko, A.Kazakov, D.Turaev – Physica D, 2017]

Let us consider system Eq. (1). By means of the coordinate change

$$\xi = \frac{\psi_1 - \psi_3}{2}, \qquad \eta = \frac{\psi_1 + \psi_3 - \pi}{2}, \qquad\qquad (3)$$
$$\rho = \frac{\psi_1 + \psi_3 - \pi}{2} + \psi_2 - \pi,$$

the system is brought to the following form

$$\begin{aligned} \dot{\xi} &= 2\varepsilon \sin\xi \sin\eta, \\ \dot{\eta} &= 1 - \varepsilon\cos(\rho - \eta) - 2\varepsilon\cos\xi\cos\eta, \\ \dot{\rho} &= 2 + \varepsilon\cos(\rho - \eta). \end{aligned} \qquad (4)$$

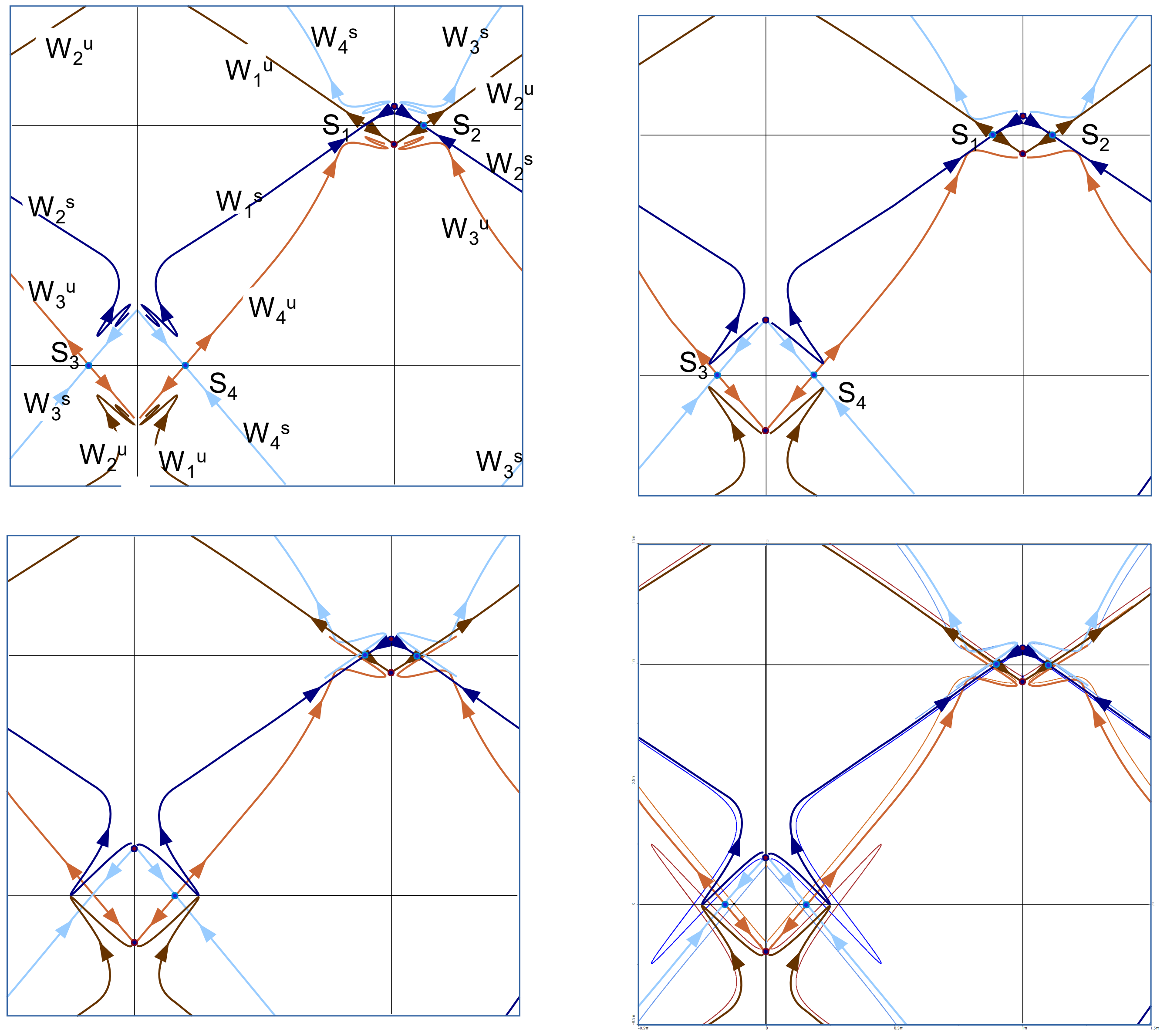
After the time change $dt_{new} = (2 + \varepsilon \cos(\rho - \eta))dt$ system (4) recasts as

$$\begin{aligned}
\dot{\xi} &= \frac{2\varepsilon \sin \xi \sin \eta}{2 + \varepsilon \cos(\rho - \eta)}, \\
\dot{\eta} &= \frac{1 - \varepsilon \cos(\rho - \eta) - 2\varepsilon \cos \xi \cos \eta}{2 + \varepsilon \cos(\rho - \eta)}, \\
\dot{\rho} &= 1,
\end{aligned}$$

i.e., a non-autonomous time-periodic system

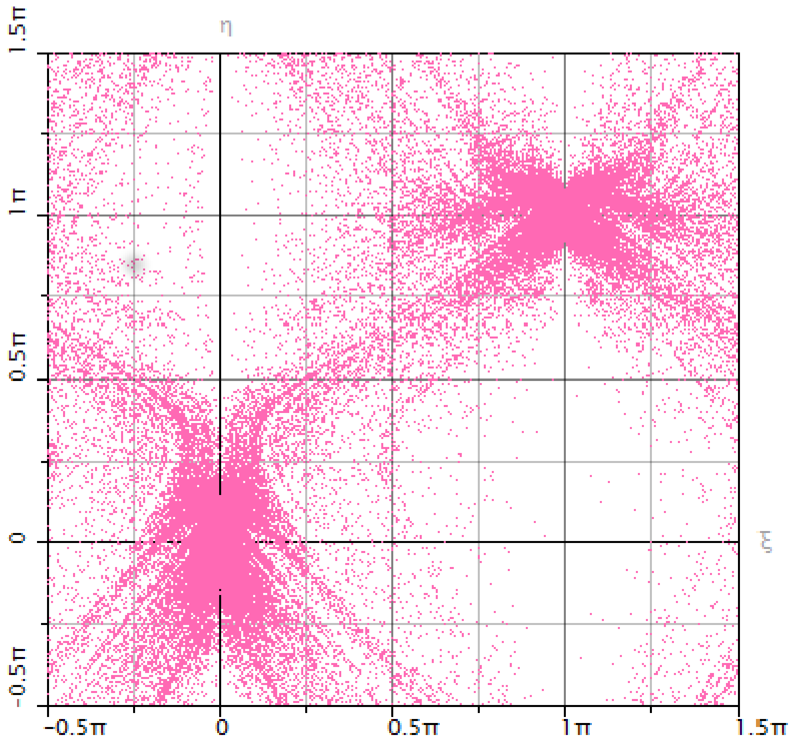
$$\begin{aligned}
\dot{\xi} &= \frac{2\varepsilon \sin \xi \sin \eta}{2 + \varepsilon \cos(t - \eta)}, \\
\dot{\eta} &= \frac{1 - \varepsilon \cos(t - \eta) - 2\varepsilon \cos \xi \cos \eta}{2 + \varepsilon \cos(t - \eta)}.
\end{aligned} \tag{5}$$

Note that system (5) is well-defined for all $\varepsilon < 2$.

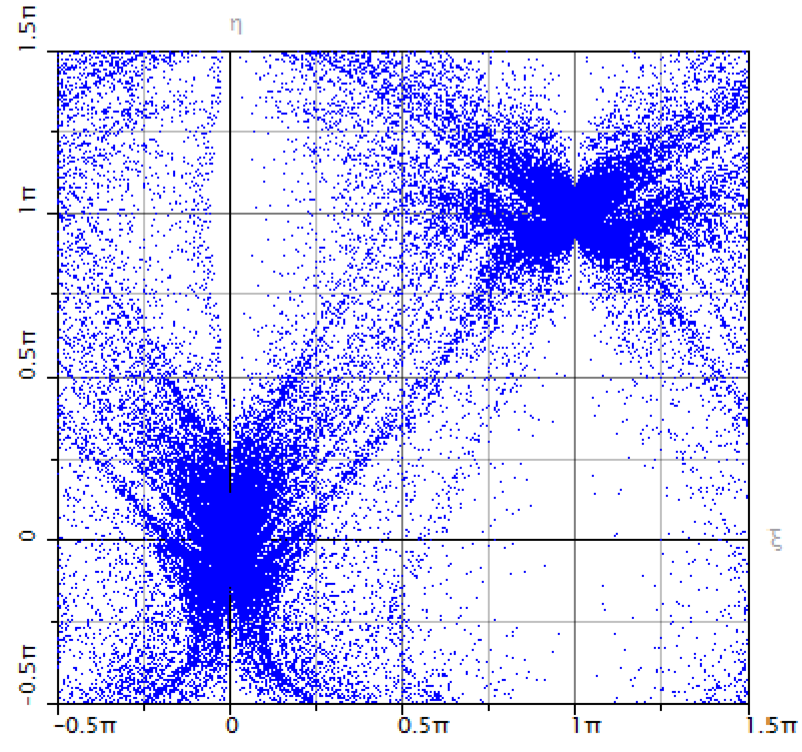


W_2^u
W_1^u
W_4^s
W_3^s
W_2^u
S_1
S_2
W_2^s
W_2^s
W_1^s
W_3^u
W_3^u
W_4^u
S_3
S_4
W_3^s
W_4^s
W_2^u
W_1^u
W_3^s
S_1
S_2
S_3
S_4

Attractor



Repeller



Example 3: (First ex. of general (non-reversible) system)

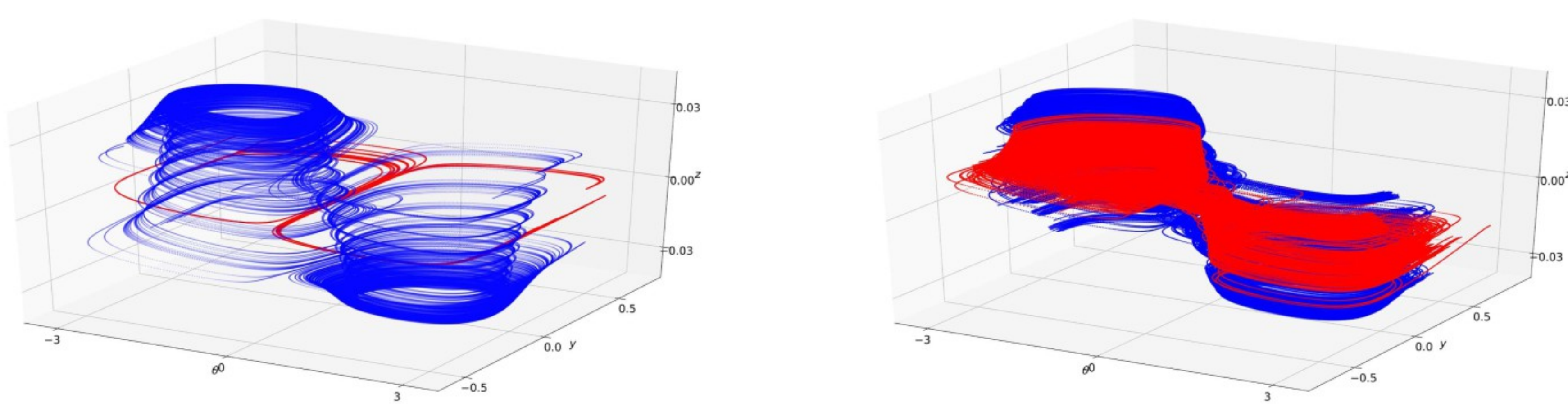
Emelyanova-Nekorkin of oscillators with adaptive couplings (CHAOS, 2019)

We consider a system of two phase oscillators interacting by adaptive couplings, where the dynamics of the phases $\{\varphi_1(t), \varphi_2(t)\}$ and the coupling weights $\{\kappa_1(t), \kappa_2(t)\}$ are given by

$$\begin{aligned} \frac{d\varphi_1}{dt} &= \omega_1 - \kappa_1 \sin(\varphi_1 - \varphi_2 + \alpha), \\ \frac{d\varphi_2}{dt} &= \omega_2 - \kappa_2 \sin(\varphi_2 - \varphi_1 + \alpha), \\ \frac{d\kappa_1}{dt} &= -\varepsilon\,(\sin(\varphi_1 - \varphi_2 + \beta) + \kappa_1), \\ \frac{d\kappa_2}{dt} &= -\varepsilon\,(\sin(\varphi_2 - \varphi_1 + \beta) + \kappa_2), \end{aligned} \tag{1}$$

where $(\varphi_1, \varphi_2) \in \mathbb{S}^2, (\kappa_1, \kappa_2) \in \mathbb{R}^2$.

The parameter α ($0 < \alpha < \frac{\pi}{2}$) characterizes the delay in signal transmission from one oscillator to another. The adaptivity rule is controlled by the parameter β ($0 < \beta < 2\pi$), which allows one to interpolate between the different adaptation modalities. In this paper, we focus on the parameter area that corresponds to the anti-Hebbian rule of adaptivity;[9] that is, the values of the coupling coefficients decrease if the phase difference of the oscillators is close to zero and increase if it is close to $\pm\pi$. The parameter ε ($\varepsilon \ll 1$) defines the scale separation between the fast dynamics of the phases and the slow dynamics of adaptation. The parameter $\gamma = \omega_1 - \omega_2$ characterizes the detuning of the natural frequencies of oscillators.



Heteroclinic cycles and intersection of attractor and repeller

- **Newhouse region** is an open (in C^r-topology, $r \geq 2$) region NR where systems with homoclinic tangencies are dense

Newhouse Theorem. NRs exist in any neighbourho od of any system with homoclinic tangency.

Proofs in• [Newhouse-79] – for dim 2;• [GST-93] – for any dimension (also in [PalisViana-94], [Romero-95]).

Definition. A Newhouse region NR_{mix} is called NR with mixed dynamics, if systems with mixed dynamics are dense (generic!) in it. (They are called also **Absolute Newhouse Regions)**

Theorem (GST97,GST99,GST07).
Let $f \in NR_{mix}$ have a closed uniformly hyperbolic invariant set Λ.
Then the following property is generic

$$\text{Closure(sinks)} \cap \text{Closure(sources)} \supset \Lambda$$

Regions *NRmix* exist

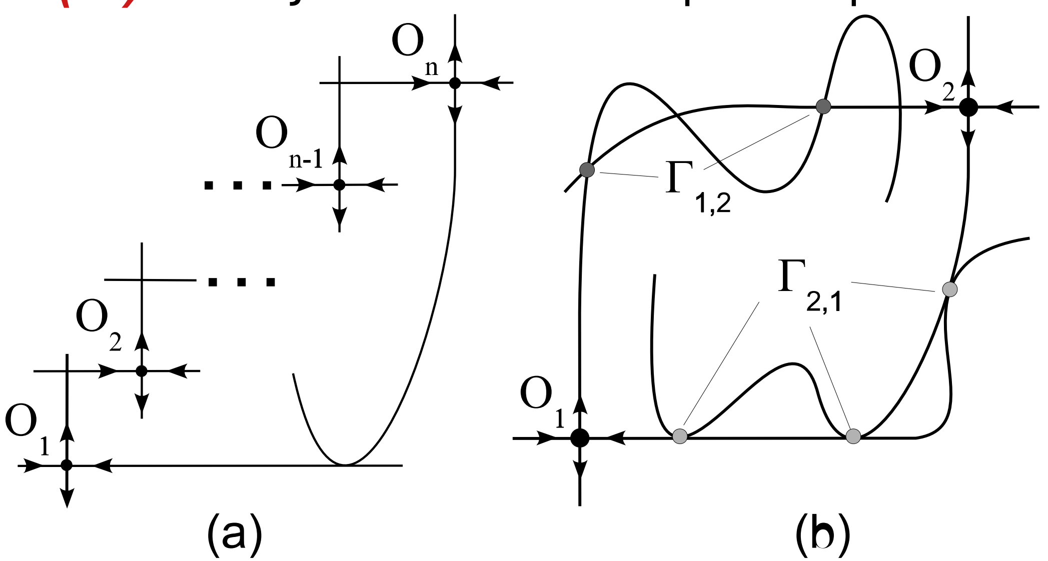
In particular, it was proved in

[1] Gonchenko, Turaev, Shilnikov *Proc. Steklov Inst. Math., 1997, v.216*

that ***NRmix*** exist near two-dimensional diffeomorphisms with nontransversal heteroclinic cycles containing at least two saddle periodic orbits O_1 and O_2 such that

$$J(O_1) < 1 \text{ and } J(O_2) > 1$$

where $J(O)$ is the Jacobian of the map at the point O.



Necessary conditions for mixed dynamics.

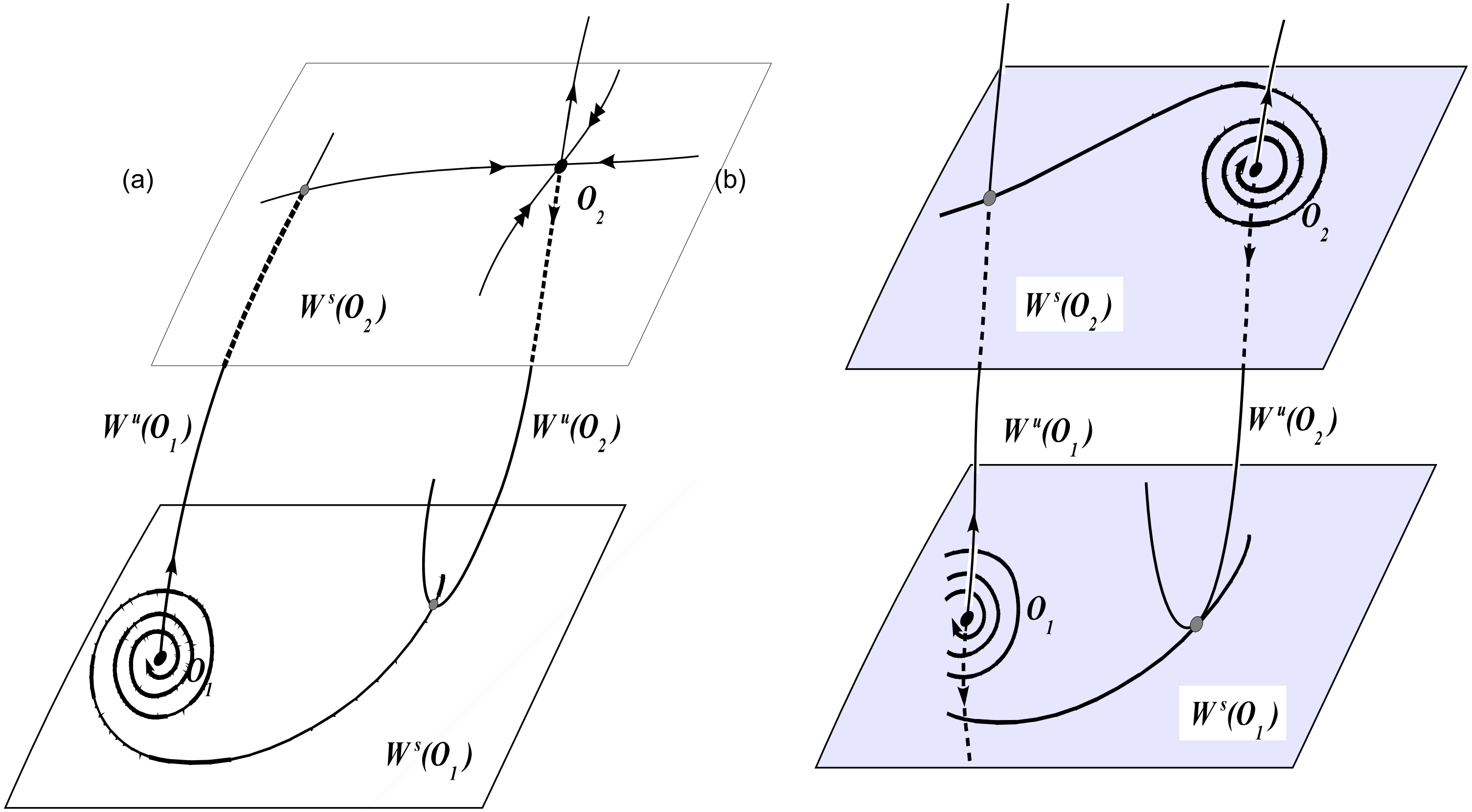
- **Both Volume contraction and expansion**

(Sign alternating divergence)

- **Complicated dynamics (Newhouse region)**
- **Absence of the partial hyperbolicity.**

This type of dynamics can exist in any dimension (Turaev,1996).

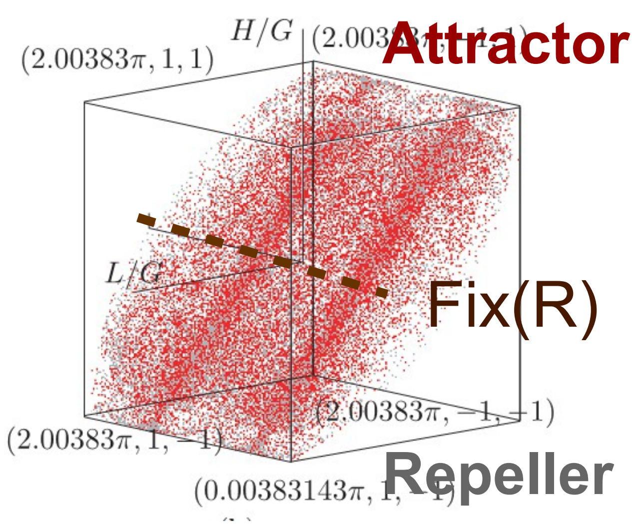
Some results for dimension 3.



It established in [GST09] and [GonOvsyannikov10], bifurcations of such heteroclinic cycles can produce Lorenz-like attractors. Moreover, Lorenz-like repellers can be also born here, [GonOvsTatjer13]. It leads to mixed dynamics involving infinitely many periodic attractors and repellers, saddles with dim $W^s = 1, 2$, stable and unstable invariant circles and strange attractors and repellers.

IIIa) Reversible Mixed Dynamics

D



A criterium of RMD

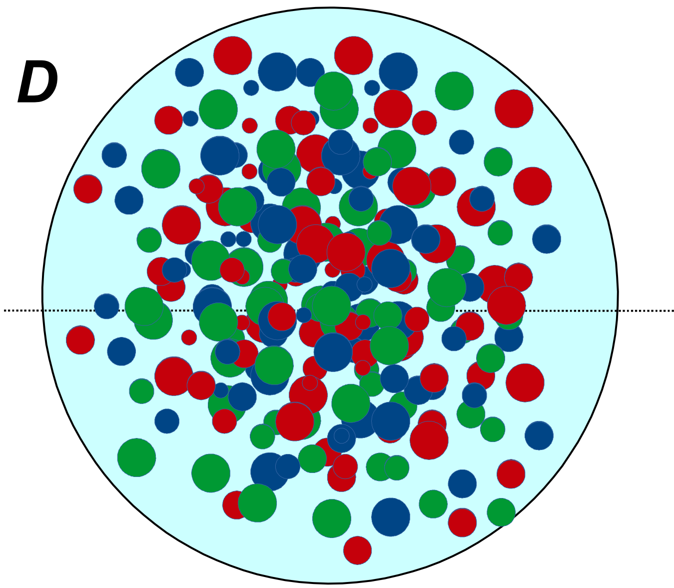
Infinitely many couples
“sink-source” +
“saddle($J<1$)”-saddle($J>1$) +
symmetric saddles($J=1$) +
symmetric **elliptic** per. orbits.

Замечание: Если dim Fix(R) = n-1

A map f is reversible if $f = R^{-1} f^{-1} R$, where R is usually an involution, i.e $R^2 = Id$. Thus, $f = R f^{-1} R$

III) Mixed Dynamics

MD as a dynamical phenomenon!



Df of MD: (одно из…)

1) coexistence of inf. many per. sinks, sourses, saddles;

2) the closures of the sets of these orbits is not empty.

MD as the third form of dynamical chaos!

$$A \cap R = RC \neq \varnothing$$

$$A \neq R$$

Main property: “Attractor” intersects with “Repeller” by a “Reversible Core” that is not empty

Moreover, RC should be quite big due to the “homoclinic tangle”.

A = RC + dissipative structures (small sinks, SA etc)
R = RC + repelling structures (small sources, strange repellers etc)

Thus, A and R are different, but, usually, they almost coincide (completely coincide in the conservative case, completely different in the dissipative dynamics).

Main question: What is attractor (repeller) ?

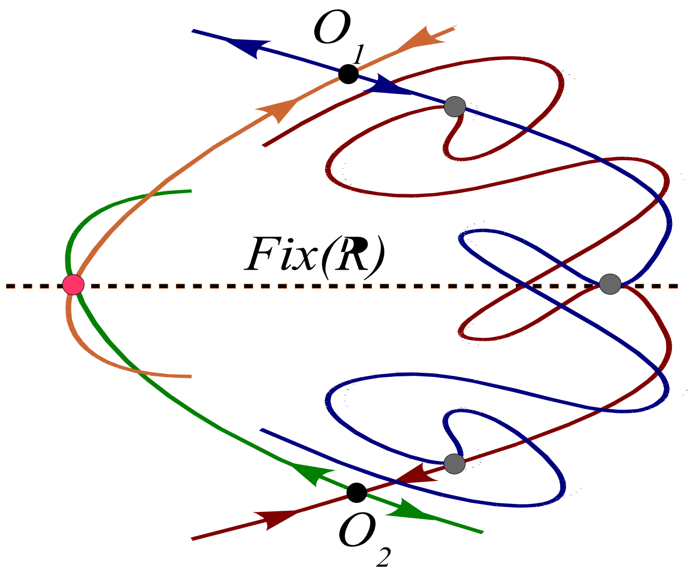
Reversible Mixed Dynamics maps.

First, it can be viewed as an universal phenomenon for reversible two-dimensional maps with complicated dynamics.

1) The contracting-expanding heteroclinic cycles are rather usual for reversible, maps (see Fig.). the map has a symmetric couple of saddle fixed points O_1 and O_2 and it is typically (general condition) when

$$J(O1) = J^{-1}(O2) < 1.$$

Thus, the phenomenon of RMD is related here to ([Lamb,Stenkin2004])



• the coexistence of infinitely many attracting, repelling, saddle and elliptic p.o.

It seems to be true that the phenomenon of mixed dynamics is universal for reversible (two-dimensional) maps with complicated dynamics when symmetric homoclinic and heteroclinic orbits are involved.

The latter can be formulated as the following **[Delshams et al 2013]**

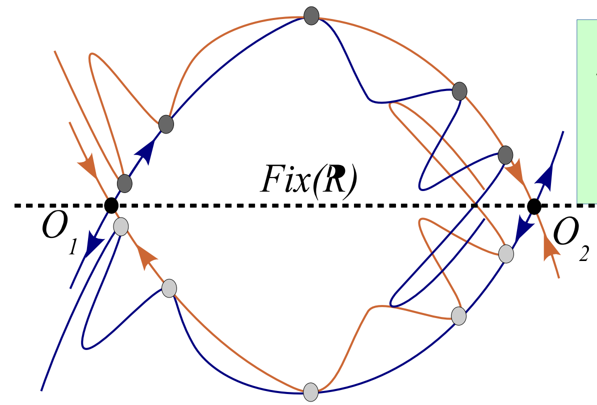
⋆ **Reversible Mixed Dynamics Conjecture.**

Reversible maps with mixed dynamics are generic in Newhouse regions of reversible maps in which there are dense maps with symmetric homoclinic tangencies or/and a couple of heteroclinic tangencies

This conjecture is true when NR in the space of C^r reversible systems are considered [GonLambRiosTuraev2014]. However, it is widely open for analytical case and for parameter families

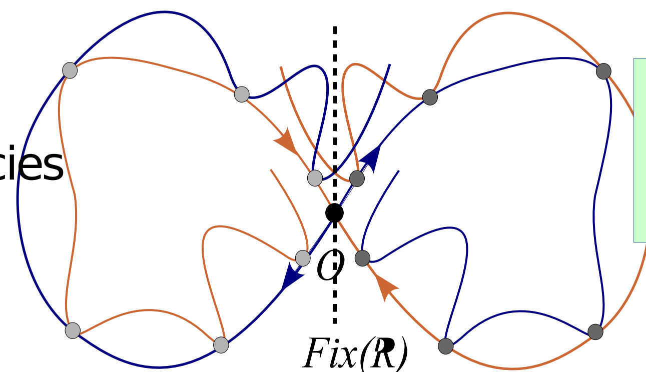
Now RMD-conjecture for 1-parameter unfoldings has been proved for two more cases.

1) an initial reversible map has 2 symmetric saddles and a symmetric couple of nontransversal heteroclinic orbits **[Delshams et al 2013]** and



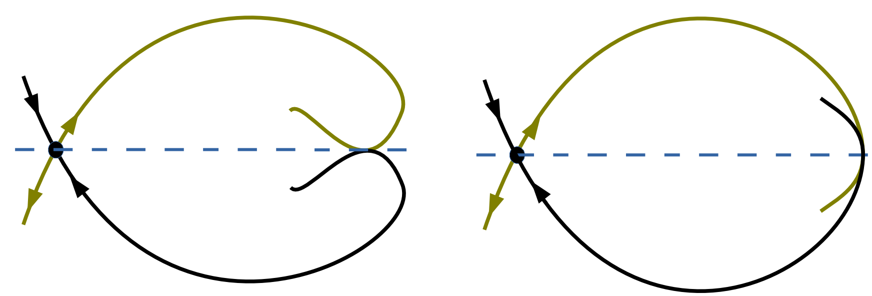
A priori conservetive case

2) an initial reversible map has a symmetric couple of homoclinic tangencies to a symmetric saddle **[Delshams et al 2015]**



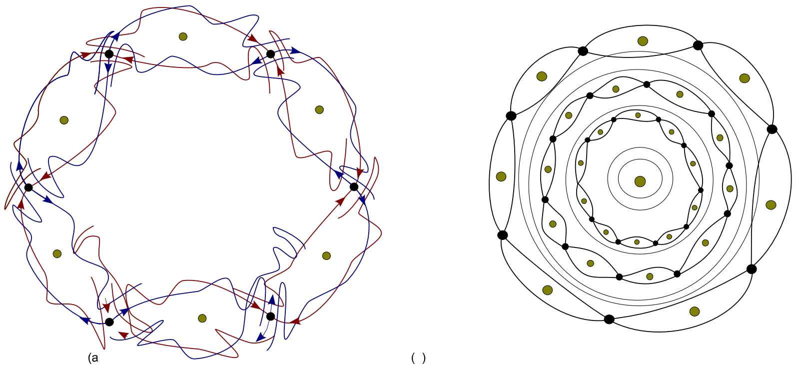
A priori non-conservetive case

However, the most interesting cases relate to the reversible maps having a symmetric saddle and a symmetric homoclinic homoclinic tangency, quadratic (a) or cubic (b). Both these case are “a priori conservative”, and a structure of *global symmetry breaking bifurcations* is not clear.



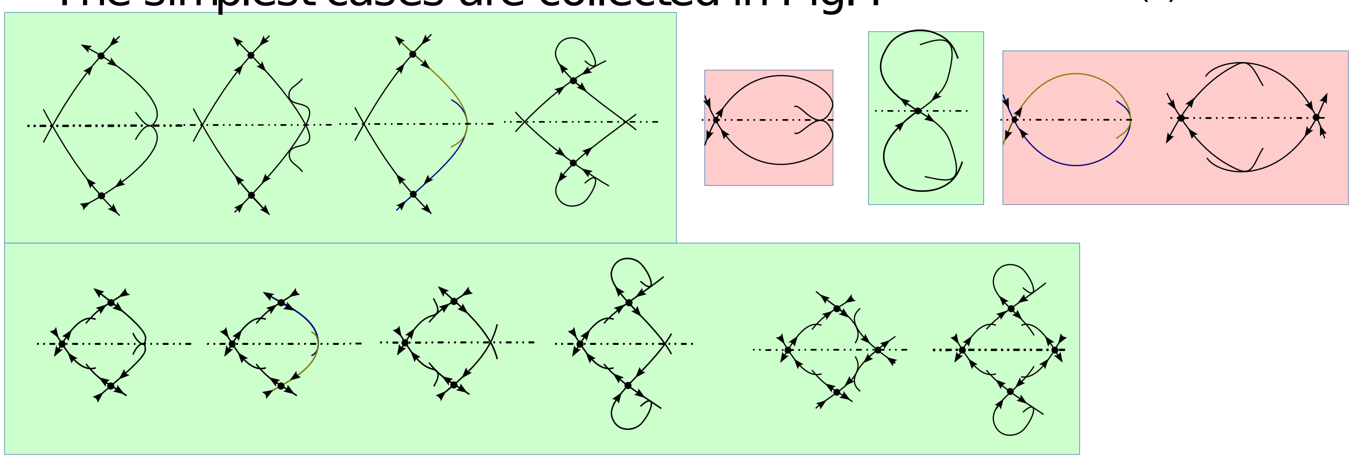
(a) (b)

Evidently, there are many various similar configurations such as
homoclinic chains of periodic orbits (a) or (b) resonant zones surrounding elliptic points etc.



(a) (b)

The simplest cases are collected in Fig.4



Generic reversible cores in two-dimensional reversible maps

Theorem 3. *All symmetric elliptic periodic orbits of a C^r-generic two-dimensional g-reversible map are reversible cores.*

g is an involution such that *dim Fix(g) =1*

Geometrical idea of the proof

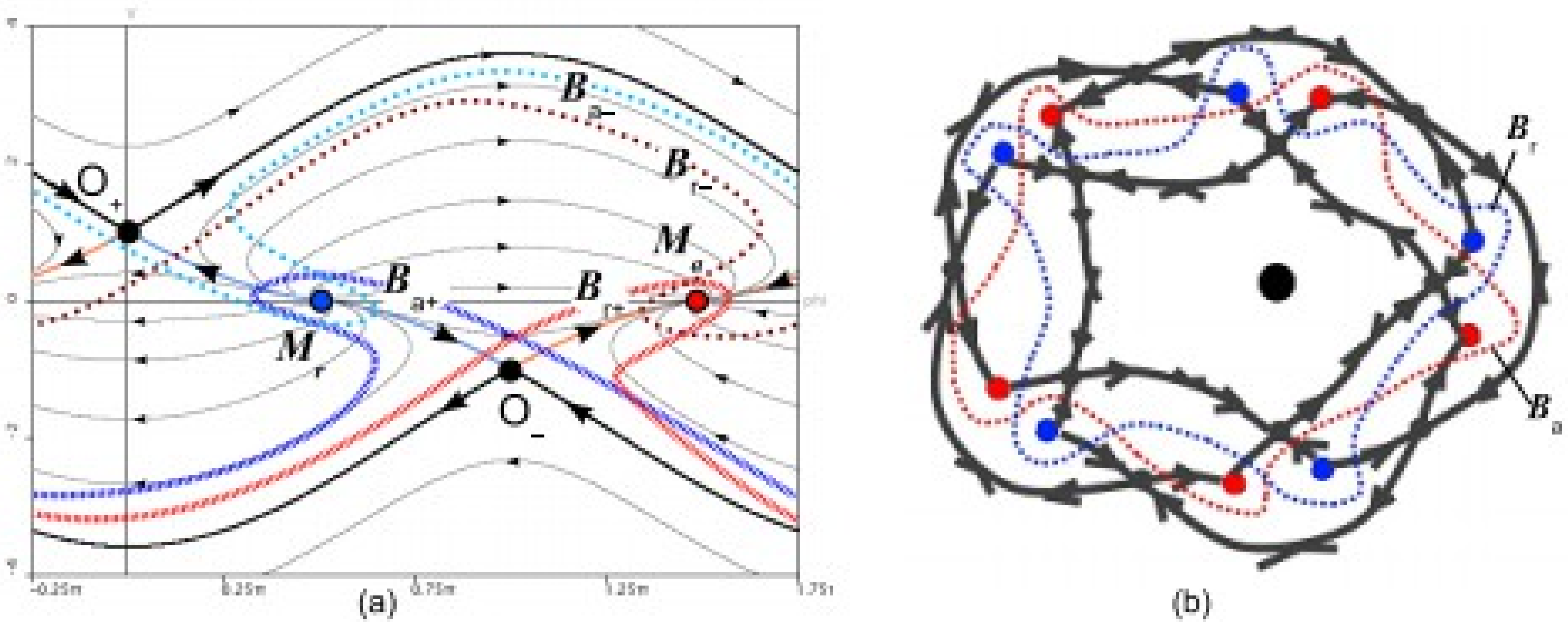


Figure 7: (a) Boundaries of absorbing domains for system (14) and its inverse (for $t \to -t$). The domains with boundaries B_{a+} and B_{r+} contain, respectively, the attractor M_a and the repeller M_r and the upper part of the cylinder. The domains with boundaries B_{a-} and B_{r-} contain, respectively, M_a and M_r and the lower part of the cylinder. (b) A pair of absorbing domains with boundaries B_a and B_r around the point $z = 0$ (for system (8) and for its time reversal).

Example: The periodically forced Duffing equation

$$\dot{x} = y, \quad \dot{y} = -x + x^3 + \varepsilon(\alpha + \beta y \sin \omega t)$$

It is reversible with respect to the change

$$x \to x,\ y \to -y,\ t \to -t$$

As the perturbation term contain y (friction) the Poincare map is not typically conservative

Reversible mixed dynamics is very universal thing.
It takes place even near elliptic points of reversible maps.Thus,
by **[Gonchenko,Lamb, Rios,Turaev, 2014], it is genrically that**

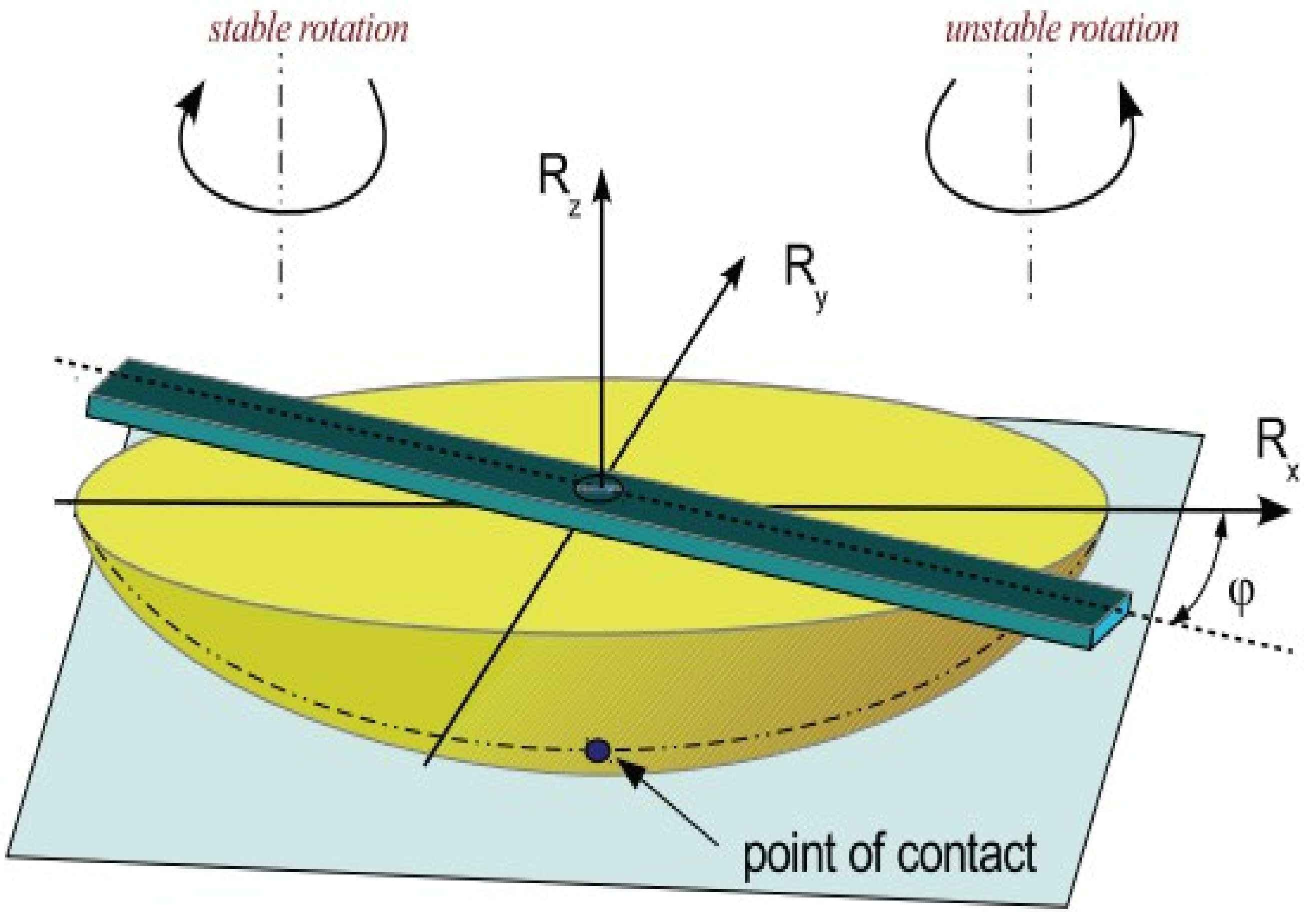
**** symmetric elliptic periodic orbit is a limit of sinks, sources and other elliptic points;***

**** symmetric elliptic periodic orbit is accumulated by symmetric Wild-hyperbolic (Newhouse) sets;***

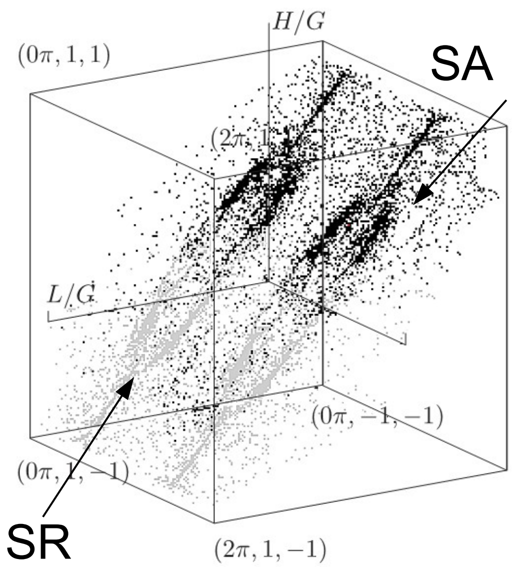
**** every point of each KAM-curve is a limit of sinks and sources.***

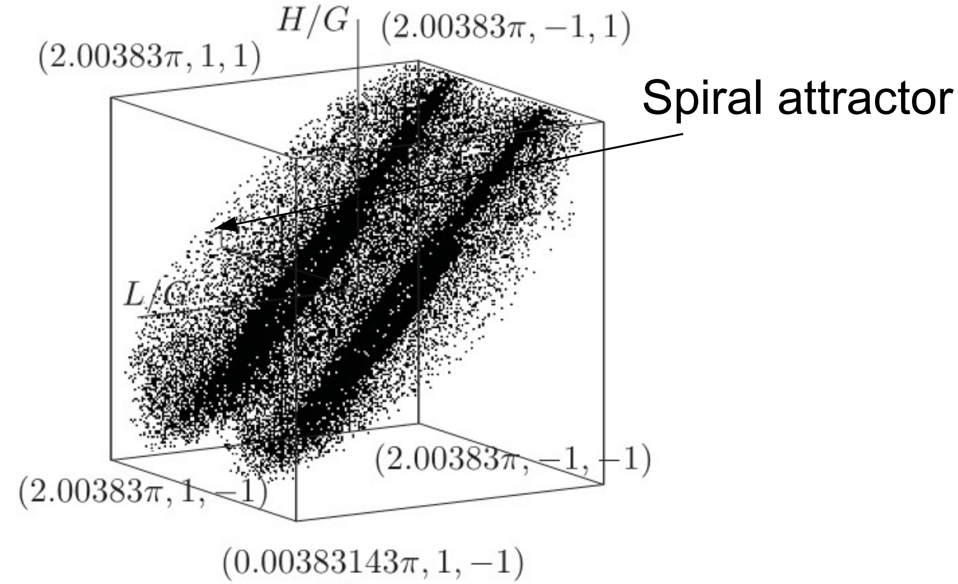
Example 2:

A nonholonomic model of a Celtic stone

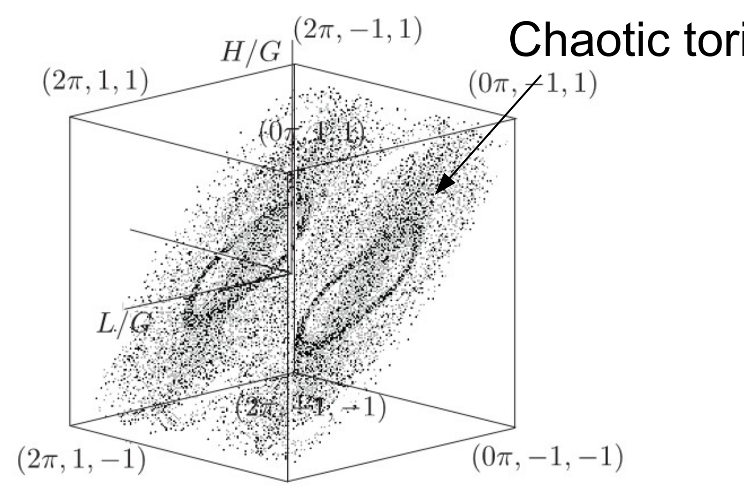


Examples of the Celtic stone chaotic dynamics





Mixed dynamics



Balthasar van der Pol
27.01.1889 – 6.10.1959
Dutch Mathematics and Physicist

Алекса́ндр Миха́йлович Ляпуно́в
— русский математик и механик.

6.06.1857 – 3.11.1918