О жизни нелинейных морских волн в пространстве Фурье

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RSF Russian Science Foundation

RFBR RUSSIAN FOUNDATION FOR BASIC RESEARCH

Introduction

1. The original motivation of this work was different

2. The effects look trivial, but ‘must see’

3. Hope to highlight some amusing features of this ‘ordinary’ stuff

4. Some introduction for the educational purpose

Introduction

Лекция С.И. Бадулина (вторник) «Азбука развития морского волнения»

Уравнения К. Хассельмана: кинетическое уравнение для ветровых волн

Уравнения В.Е. Захарова: гамильтонов формализм и динамические уравнения на нелинейные **квартеты** или пятерки

Условие **нелинейного** резонанса четверки волн

$$\mathbf{k}_1 + \mathbf{k}_2 = \mathbf{k}_3 + \mathbf{k}_4$$

$$\omega_1 + \omega_2 = \omega_3 + \omega_4$$

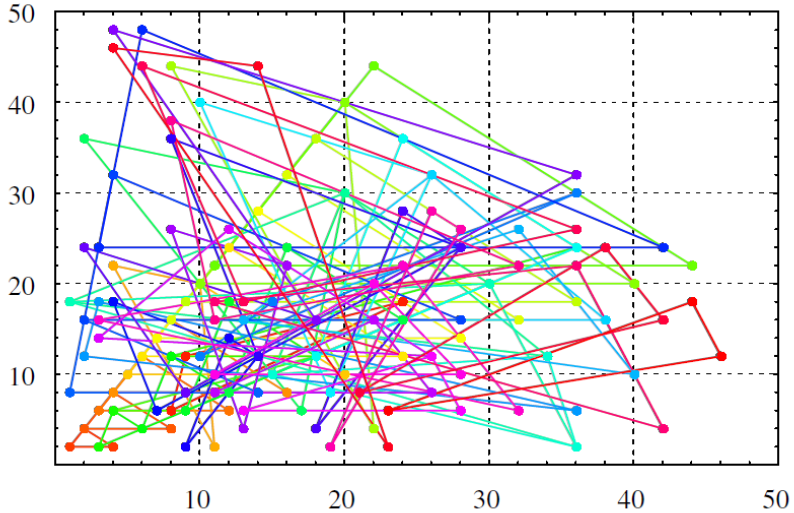
$$\omega = \omega(\mathbf{k})$$

Форма дисперсионного соотношения определяет, какие **взаимодействия возможны** (степень нелинейности, геометрия квартетов в области волновых векторов).

Коэффициент нелинейного взаимодействия (интеграл столкновений для кинетической теории) определяет **эффективность взаимодействия** (может обращаться в ноль).
Dyachenko et al (2016): (планарные волны на глубокой воде): если все волны изначально бегут вправо, то волны влево **не** возникнут.

Introduction

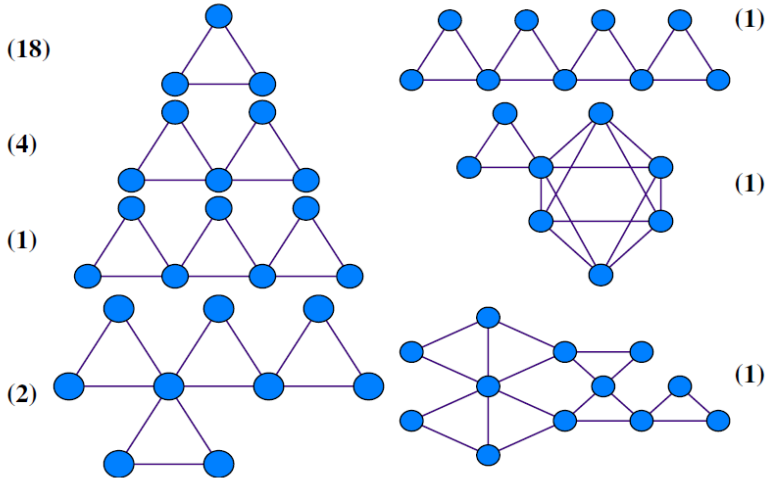
Пример трехволновых резонансов на **дискретной сетке k**



$$\mathbf{k}_1 + \mathbf{k}_2 = \mathbf{k}_3, \quad \mathbf{k} = (k_x, k_y)$$

$$\omega_1 + \omega_2 = \omega_3$$

Example of geometrical structure, spectral domain $|\mathbf{k}| < 50$

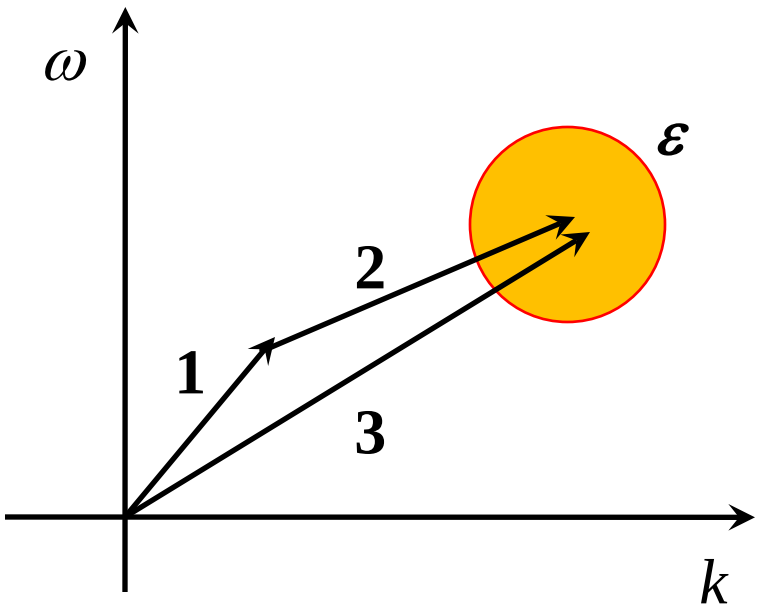


Example of topological structure, the same spectral domain. The number in brackets shows how many times corresponding cluster appears in the chosen spectral domain

[Kartashova & Mayrhofer, 2007]

Introduction

Роль **нерезонансных** взаимодействий



$$\mathbf{k}_1 + \mathbf{k}_2 = \mathbf{k}_3$$

$$\omega_1 + \omega_2 \approx \omega_3$$

$$(k_1, \omega_1) + (k_2, \omega_2) \approx (k_3, \omega_3)$$

Stiassnie & Shemer (2005): наиболее эффективное взаимодействие
НЕ при точном выполнении резонансных условий

Annenkov & Shrira (2006): «выколотые» точные резонансы
НЕ изменяют динамику существенно

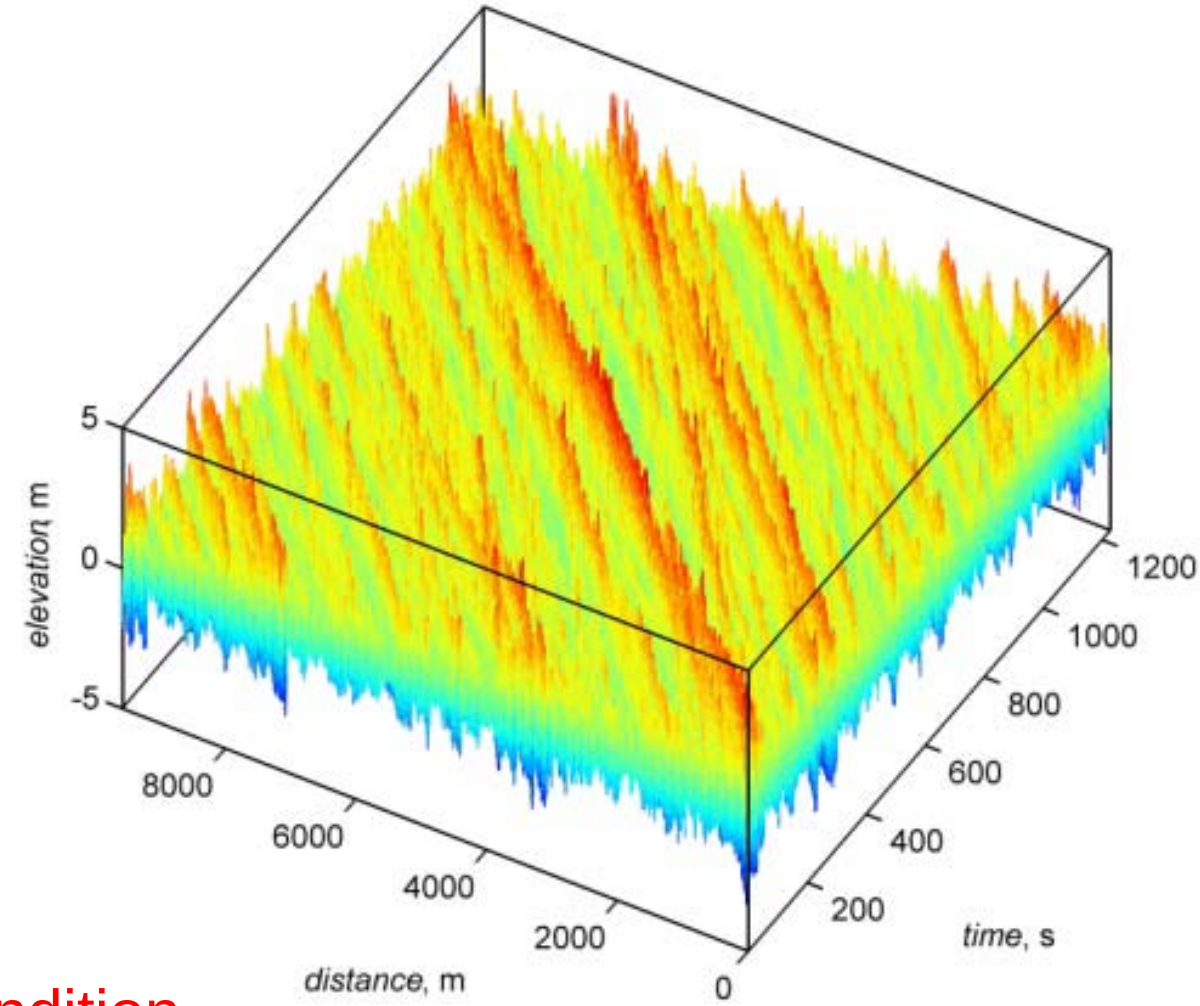
Рецепт численного моделирования: ε - область вместо δ - функции

Contents

1. The toolkit & Linear problem

Fourier transform for linear waves

The toolkit

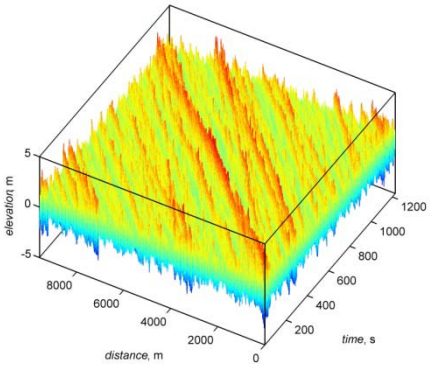


A real-valued field
in a space-time domain
(the deep-water surface displacement caused by collinear gravity waves)

The **periodic** spatial condition
The data **is not periodic** in time

Fourier transform for linear waves

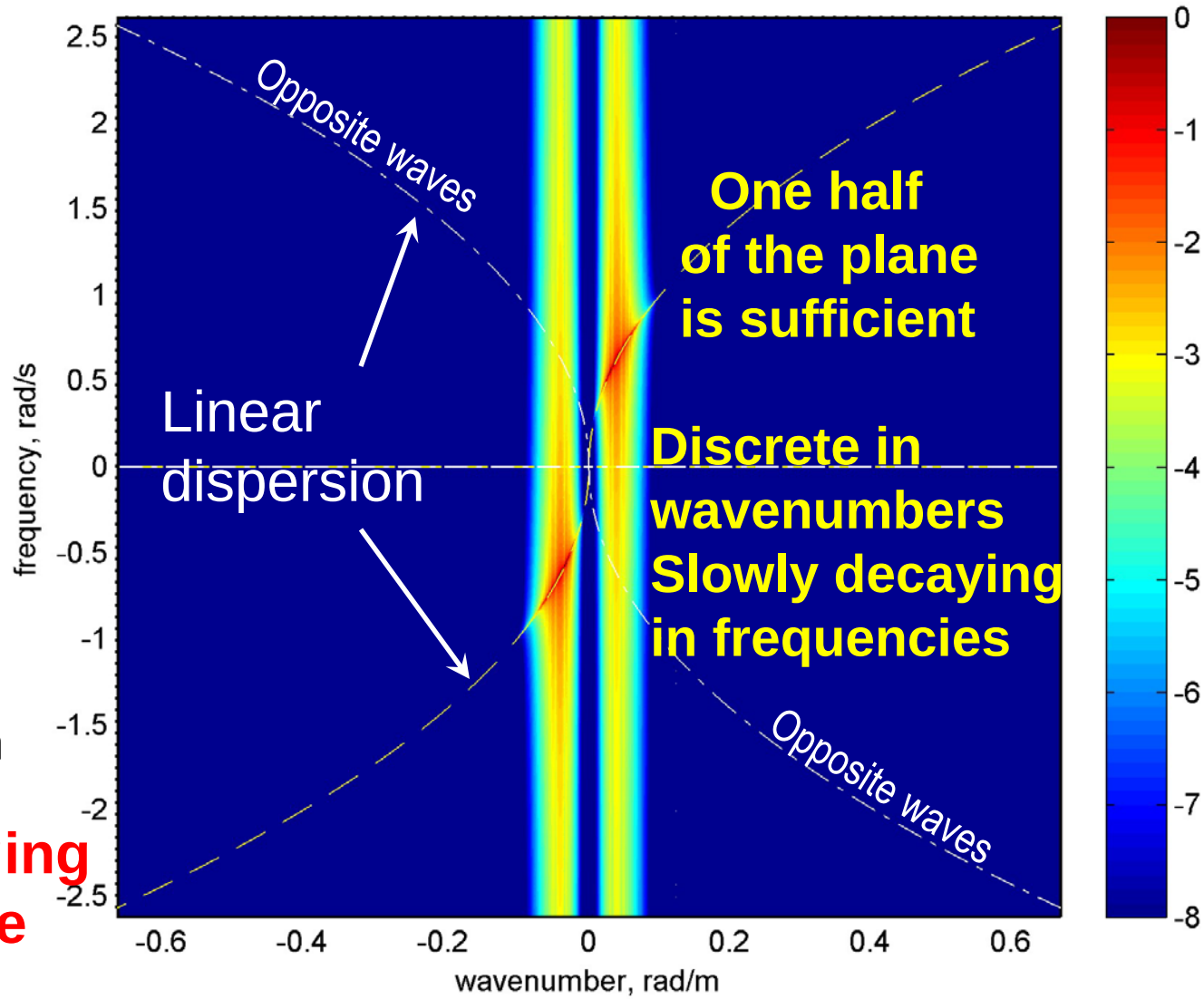
The toolkit



$$\log_{10}\sqrt{\frac{S(\omega,k)}{\max S(\omega,k)}}$$

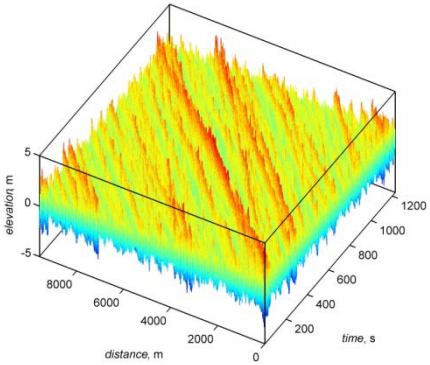
Normalized amplitudes of the *x-t* Fourier transform of the water surface elevation

Some time averaging always takes place



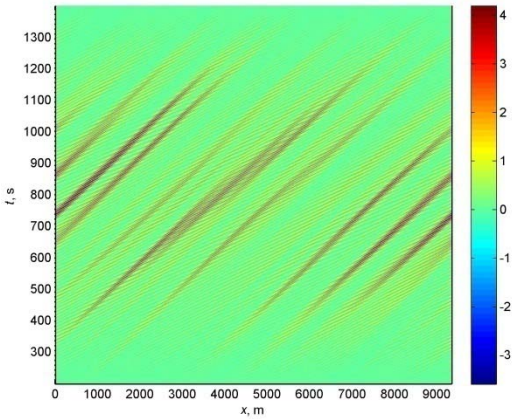
Fourier transform for linear waves

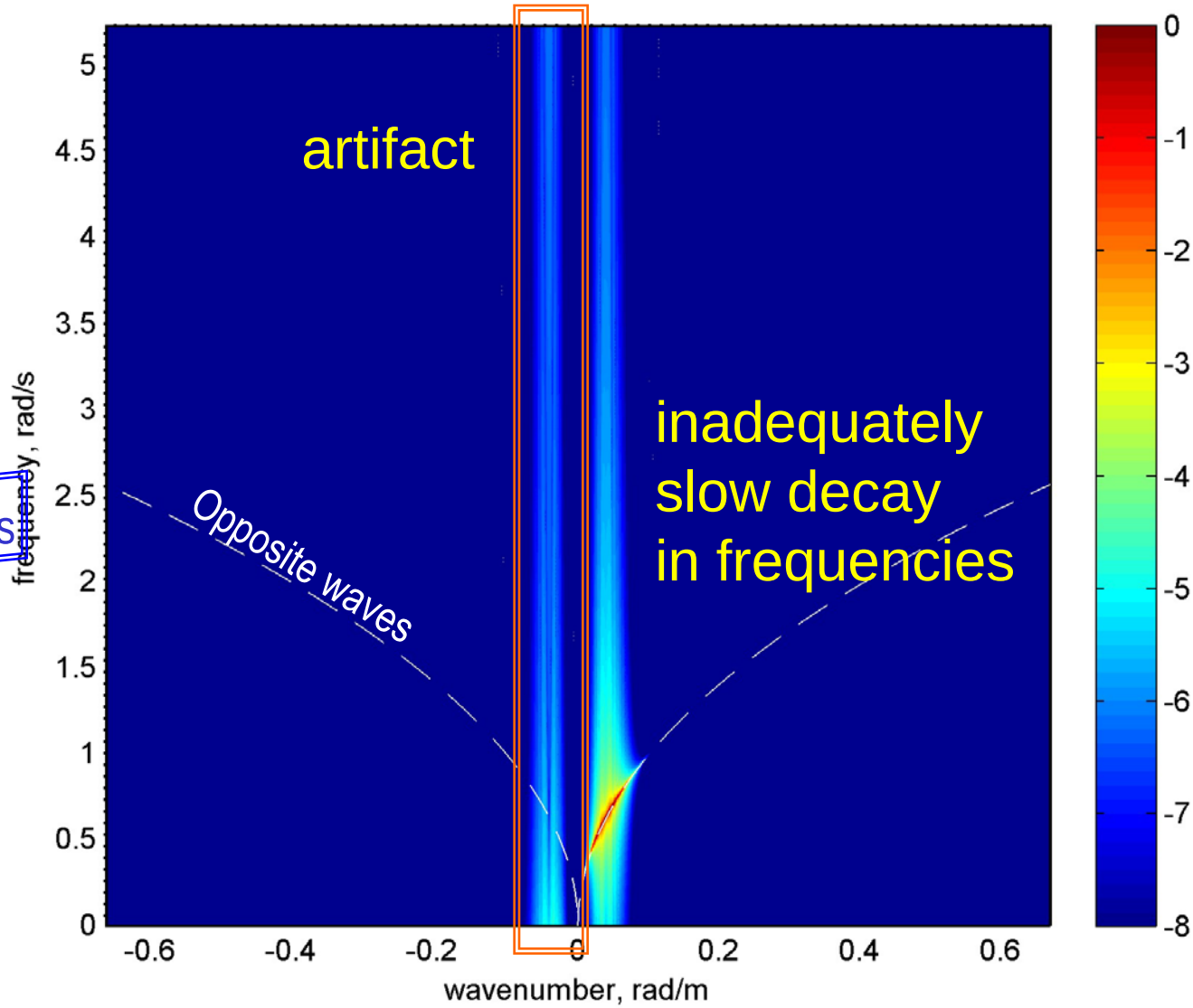
The toolkit



$$M(\tau)=\frac{1}{2}-\frac{1}{2}\cos\frac{(t-\tau)}{\pi T}, \qquad t\in[0,T)$$

Hanning data window applies



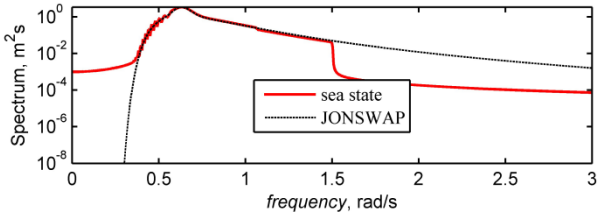


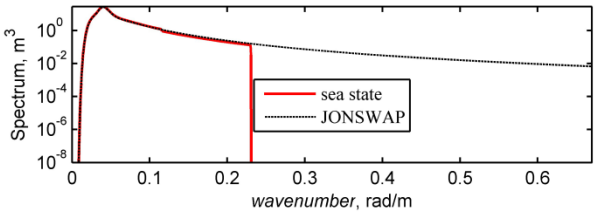
Fourier transform for linear waves

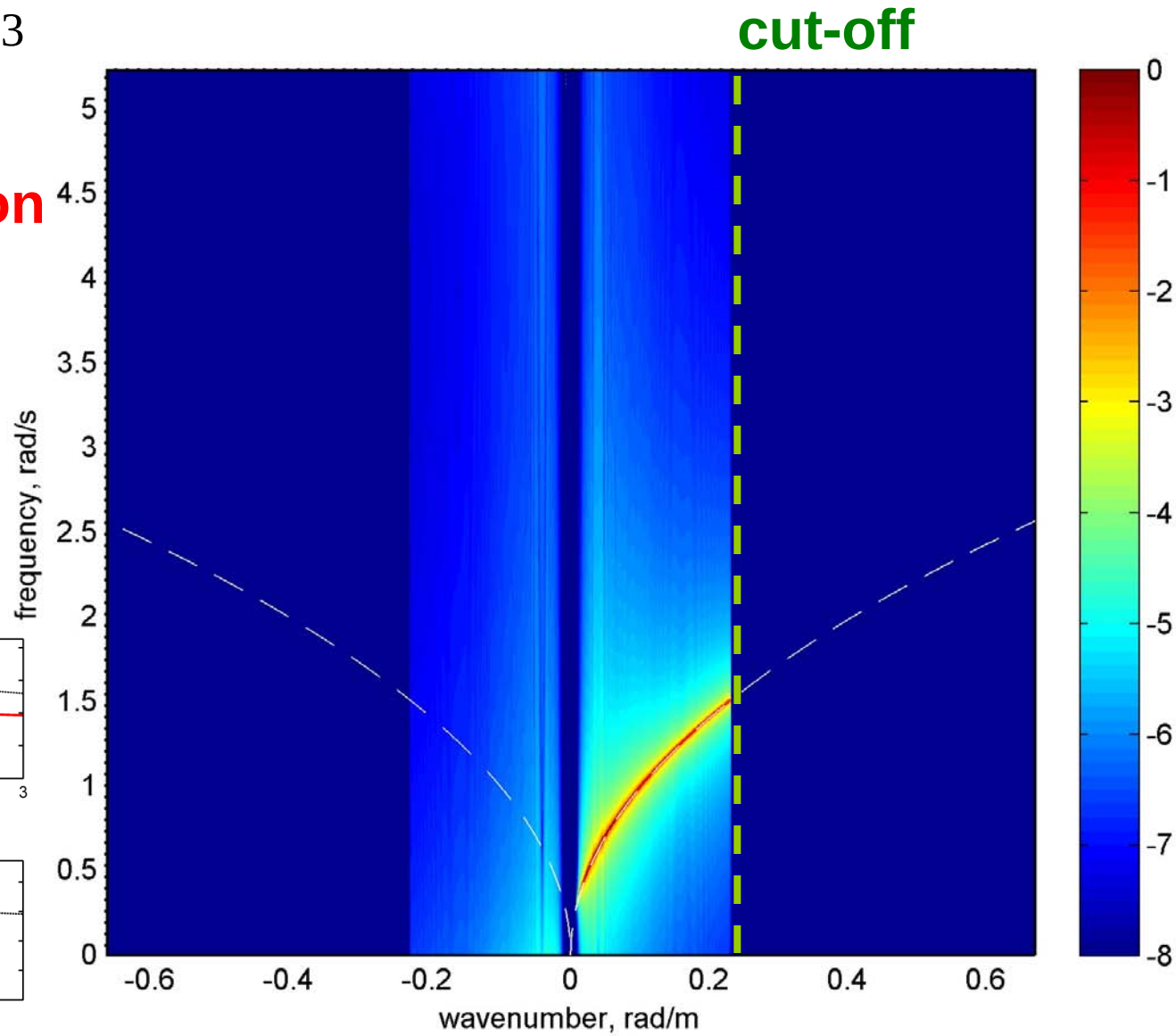
JONSWAP spectrum, the linear simulation

$H_s = 3.3$ m, $T_p = 10$ s, $\gamma = 3$

The energy location follows the linear dispersion curve







2. One nonlinear wave

Fourier transform of a nonlinear wave

A sinusoidal wavetrain, primitive potential hydrodyn. eqs.

Initial condition:

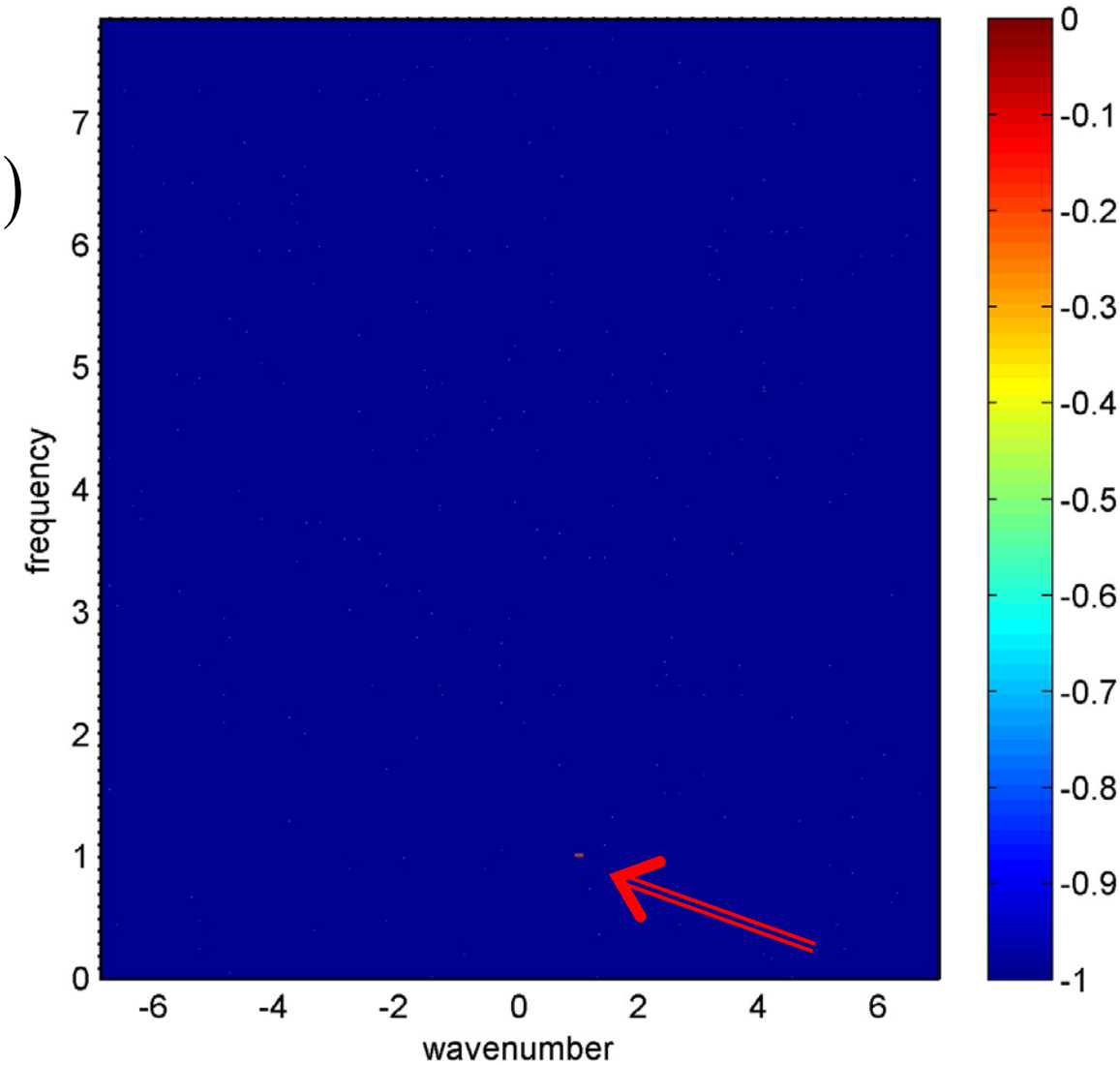
$$\eta(x,0) = A\cos(k_0 x)$$

$$\Phi^s(x,0) = AC_{ph}\sin(k_0 x)$$

$k_0 = 1$, $\varepsilon \equiv Ak_0 = 0.1$, $g = 1$

$$\log_{10}\sqrt{\frac{S(\omega,k)}{\max S(\omega,k)}}$$

Amplitudes to order ε^1 are shown



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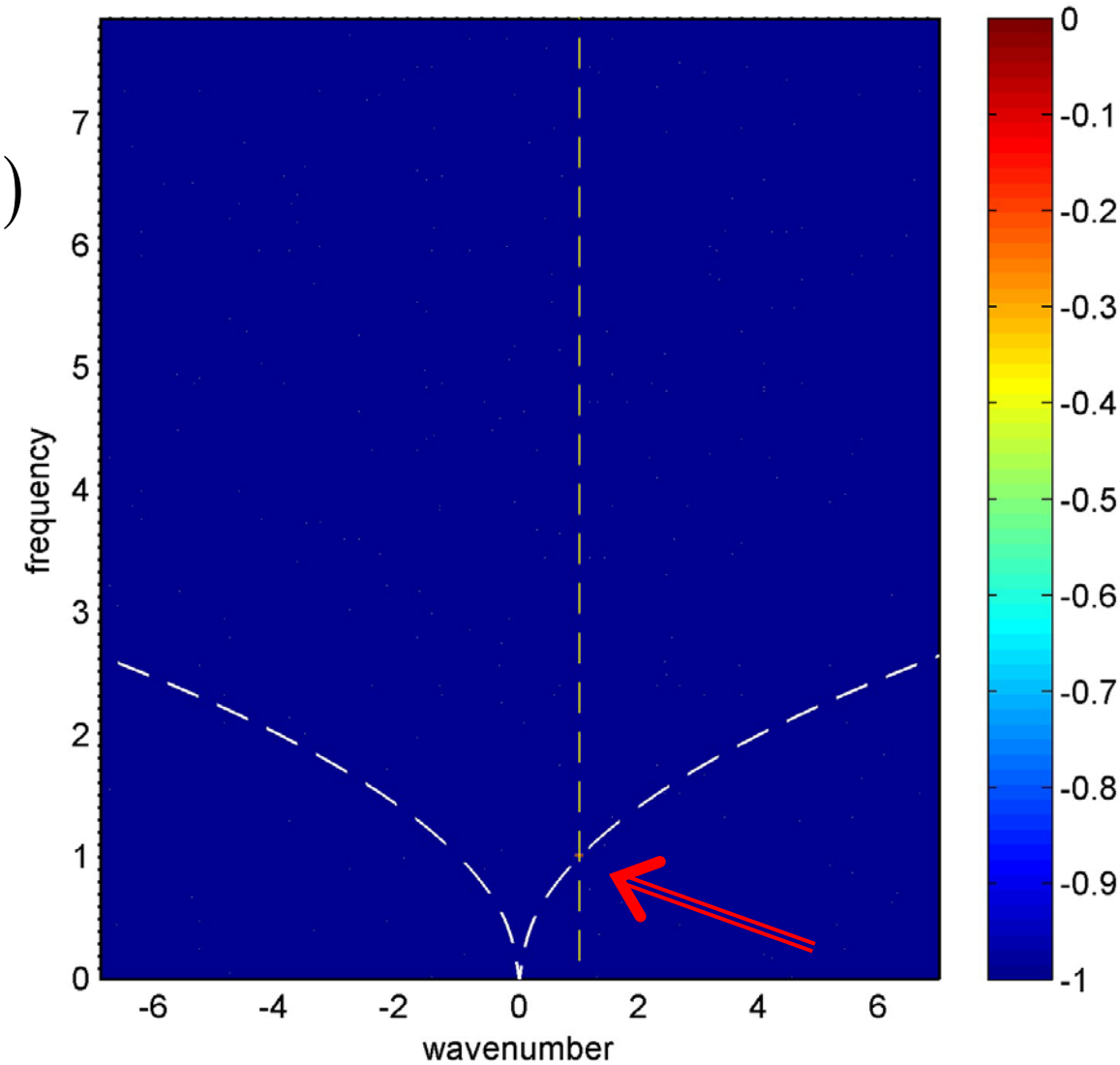

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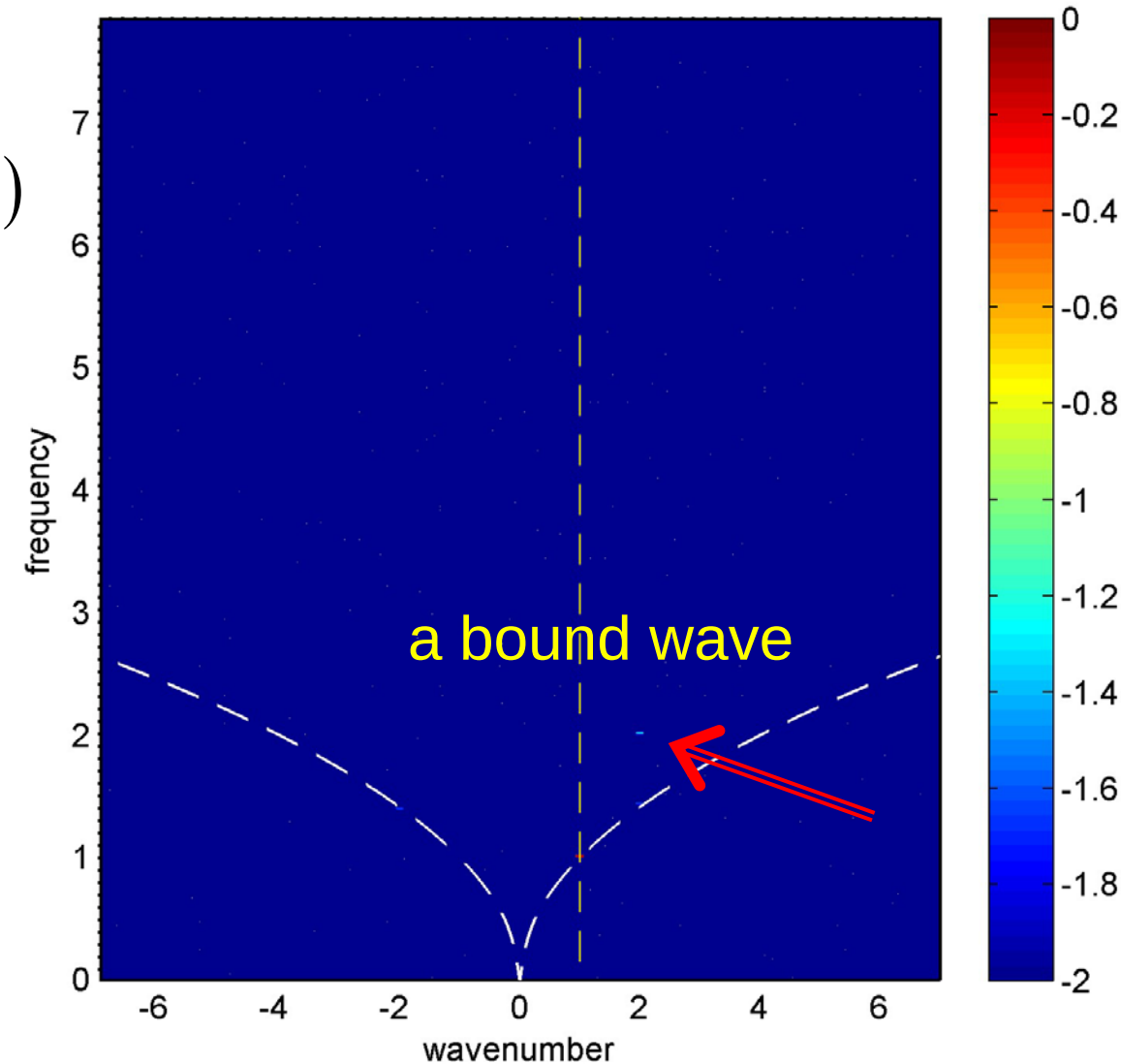
Amplitudes to order ε^2 are shown

The second harmonic $\sim \varepsilon^2$



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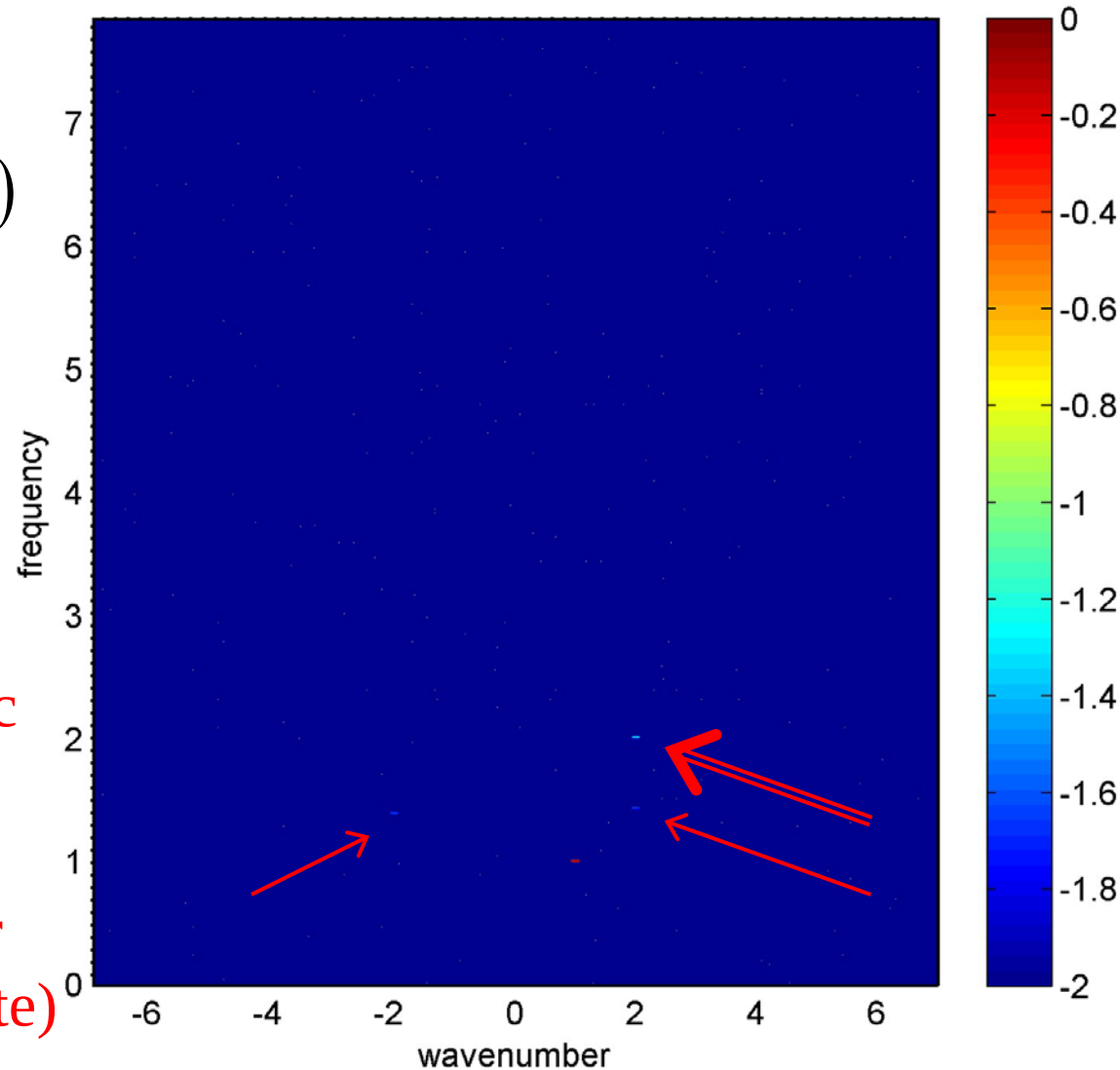
Amplitudes to order ε^2 are shown

The second harmonic $\sim \varepsilon^2$

+ free waves at the double wavenumber (following and opposite)



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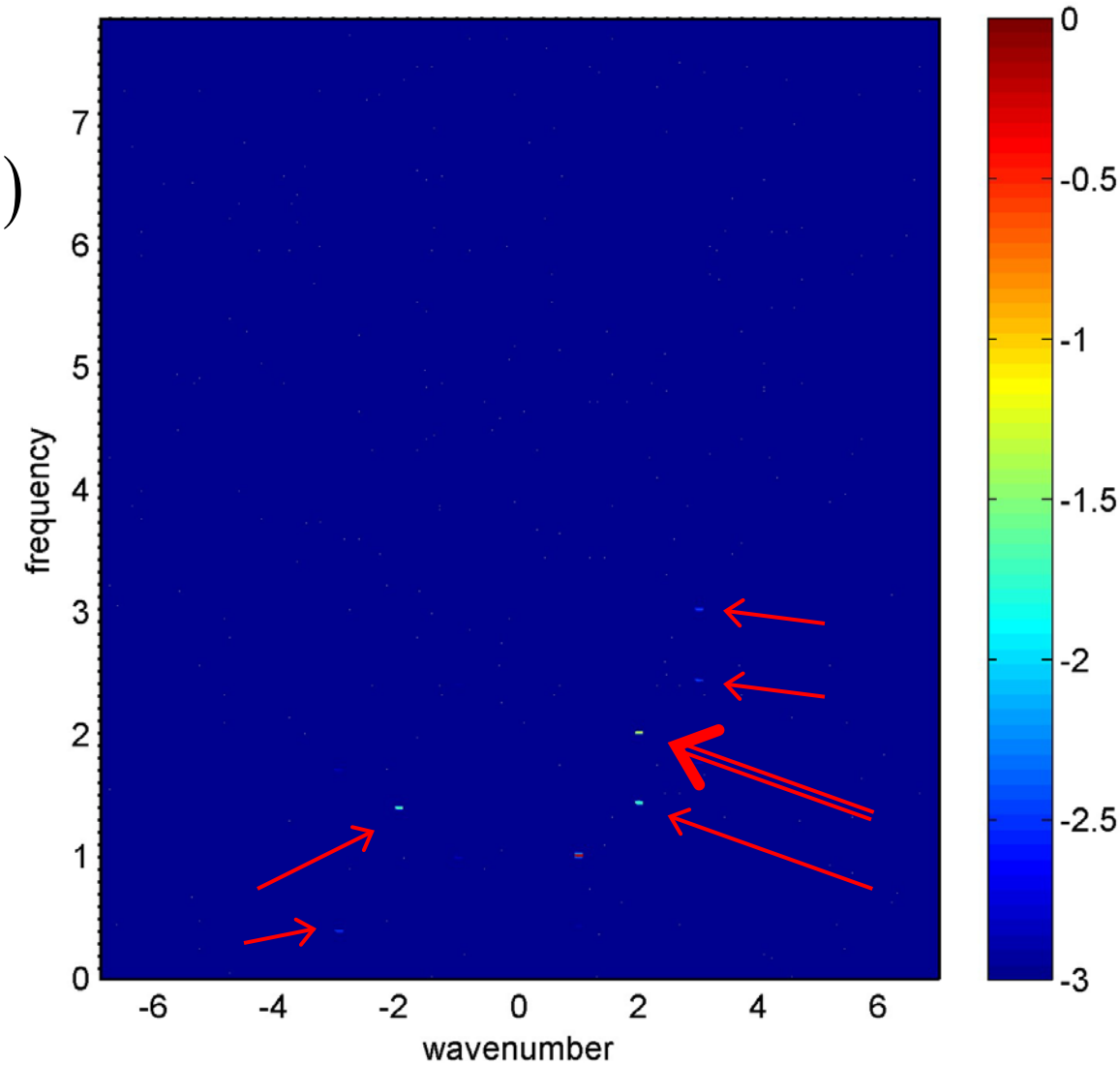
Amplitudes to order ε^3 are shown



Fourier transform of a nonlinear wave

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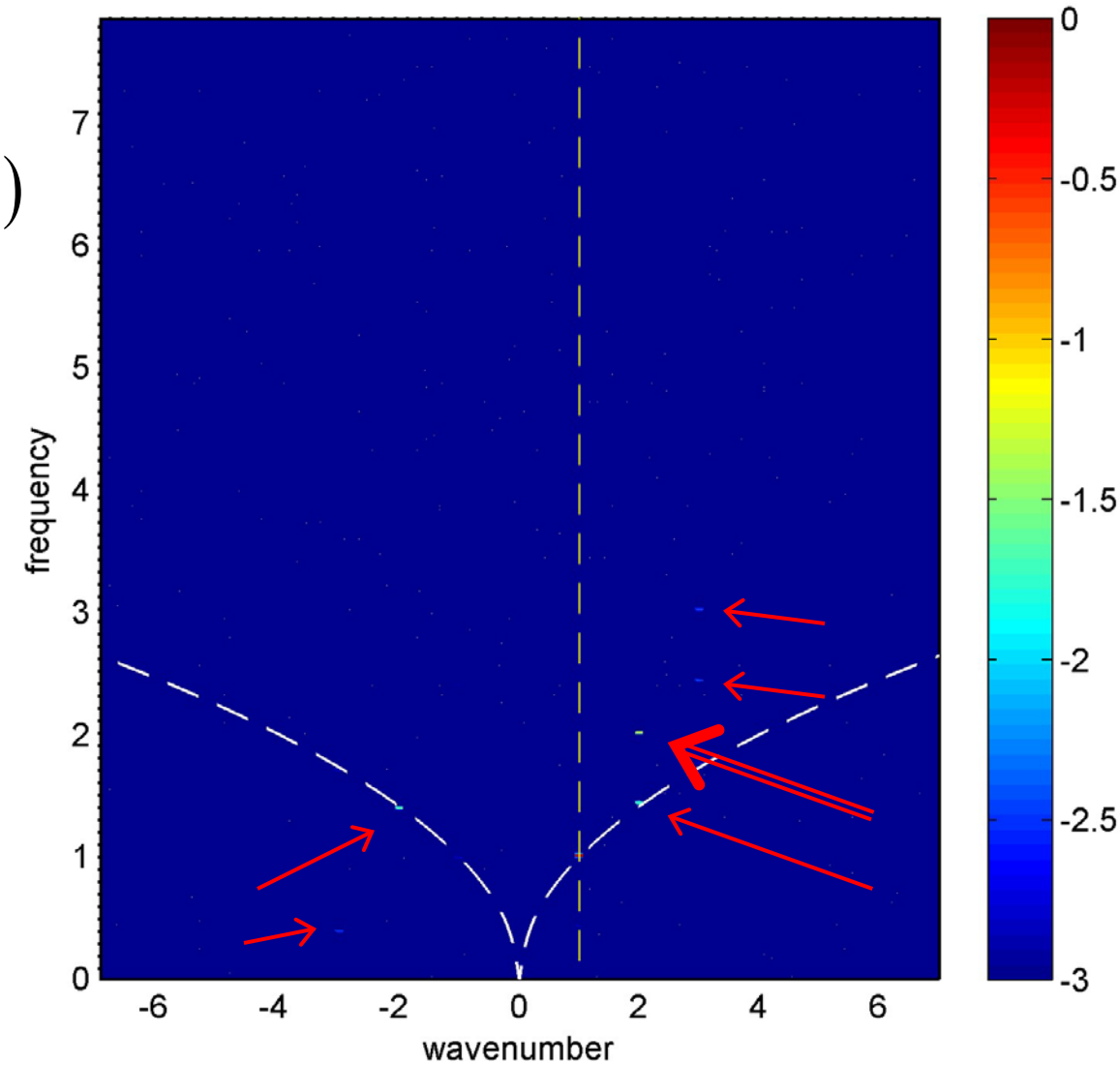

Fourier transform of a nonlinear wave

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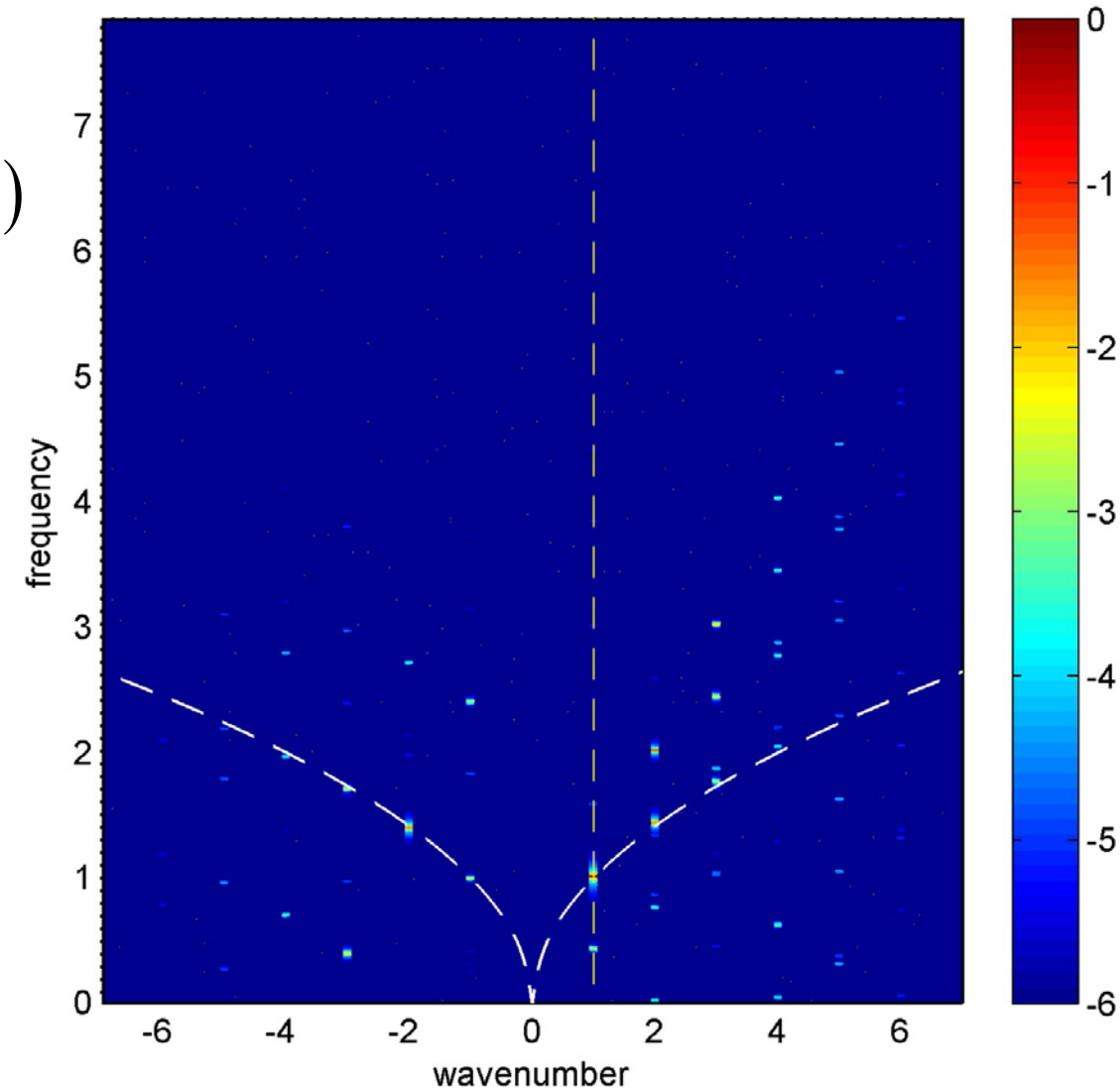
Amplitudes to order ε^6 are shown



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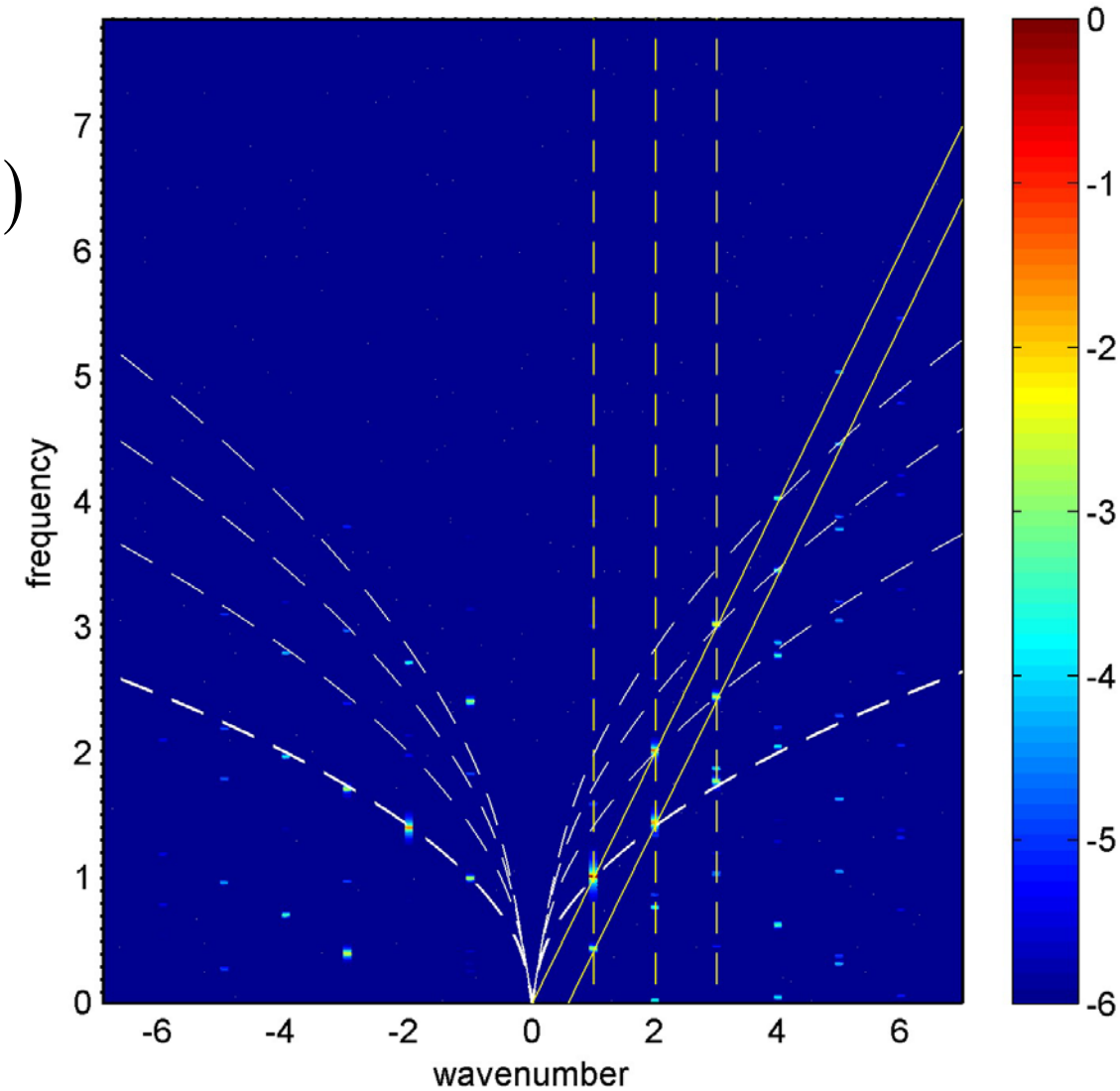
$k_0 = 1$, $\varepsilon \equiv Ak_0 = 0.1$, $g = 1$

Amplitudes to order ε^6 are shown

The phase-locked (bound) waves are located at $(nk_0, n\omega_0)$ – for the primary and secondary waves



Fourier transform of a nonlinear wave

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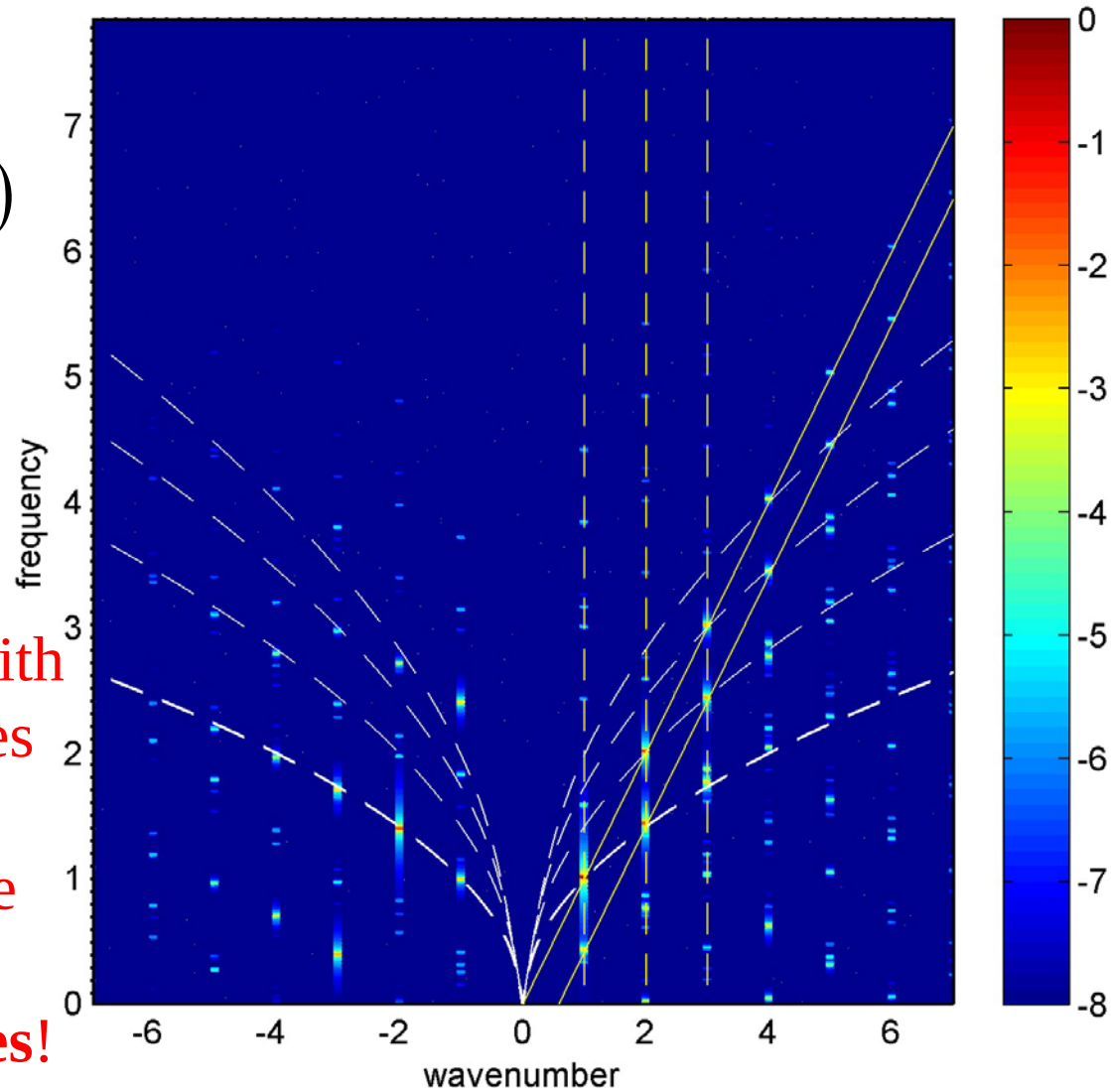
$$\Phi^s(x,0) = AC_{ph}\sin(k_0 x)$$

$k_0 = 1$, $\varepsilon \equiv Ak_0 = 0.1$, $g = 1$

Amplitudes to order ε^8 are shown

A uniform sin wave with given length generates plenty of unwanted energetic spots in the Fourier domain – **free and bound waves**!

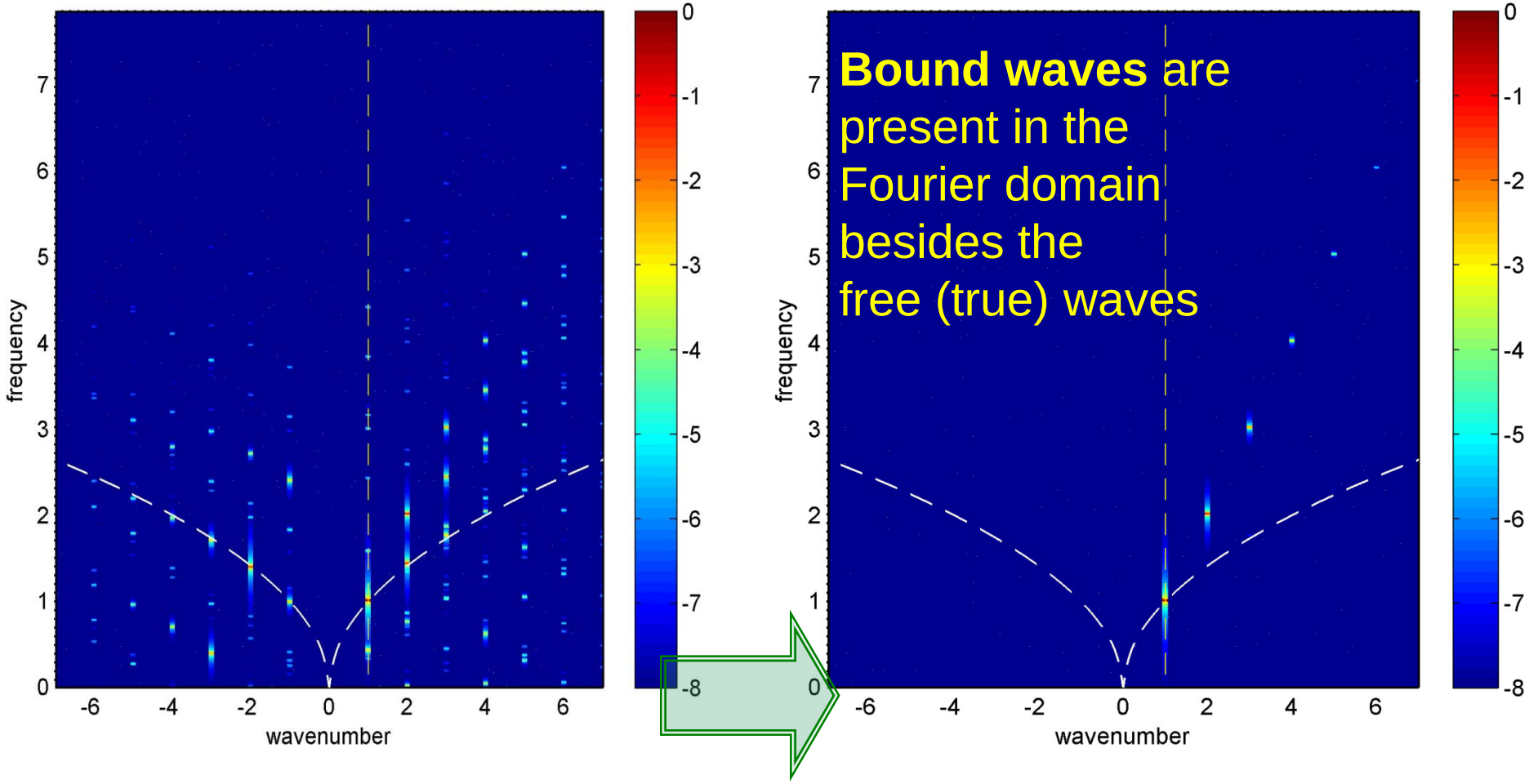


Fourier transform of a nonlinear wave

Generation of ‘true’ nonlinear waves – the Stokes waves

Sinusoidal initial condition

Stokes wave

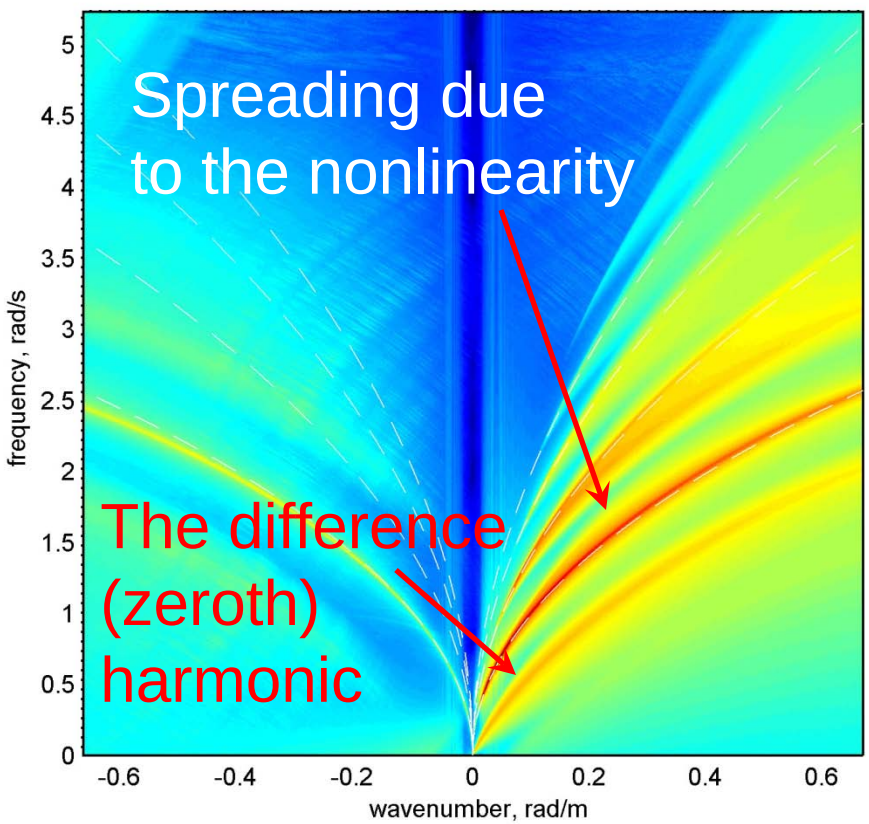


D.Dommermuth (2000) The initialization of nonlinear waves using an adjustment scheme. Wave Motion 32, 307-317: nonlinear terms are enforced slowly

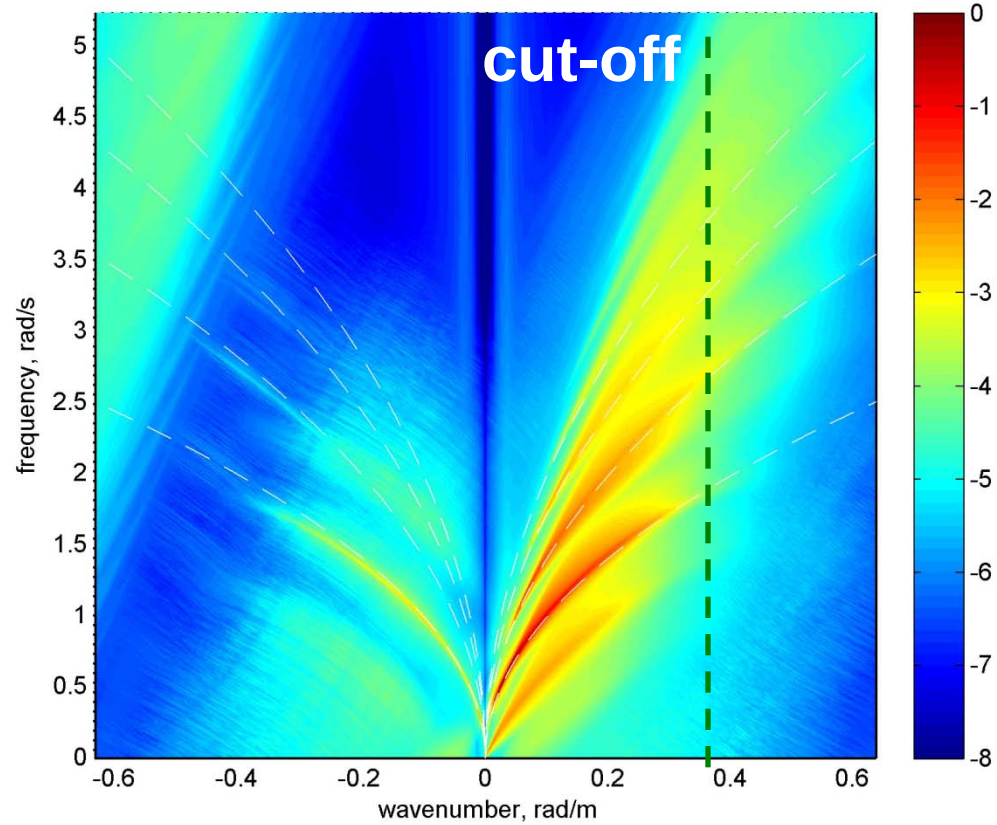
Fourier transform of irregular waves

JONSWAP irregular waves, primitive potential hydrodyn. eqs.

$\boldsymbol{H_s}$ **= 3.3 m**, T_p = 10 s, γ = 3, ε_s = 0.07



$\boldsymbol{H_s}$ **= 7 m**, T_p = 10.5 s, γ = 3.3, ε_s = 0.13



The spectral lobes responsible for free and bound waves are well separated, though they overlap in either spatial or frequency transforms !

3. Wave-group linear resonances

Fourier transform of envelope solitons

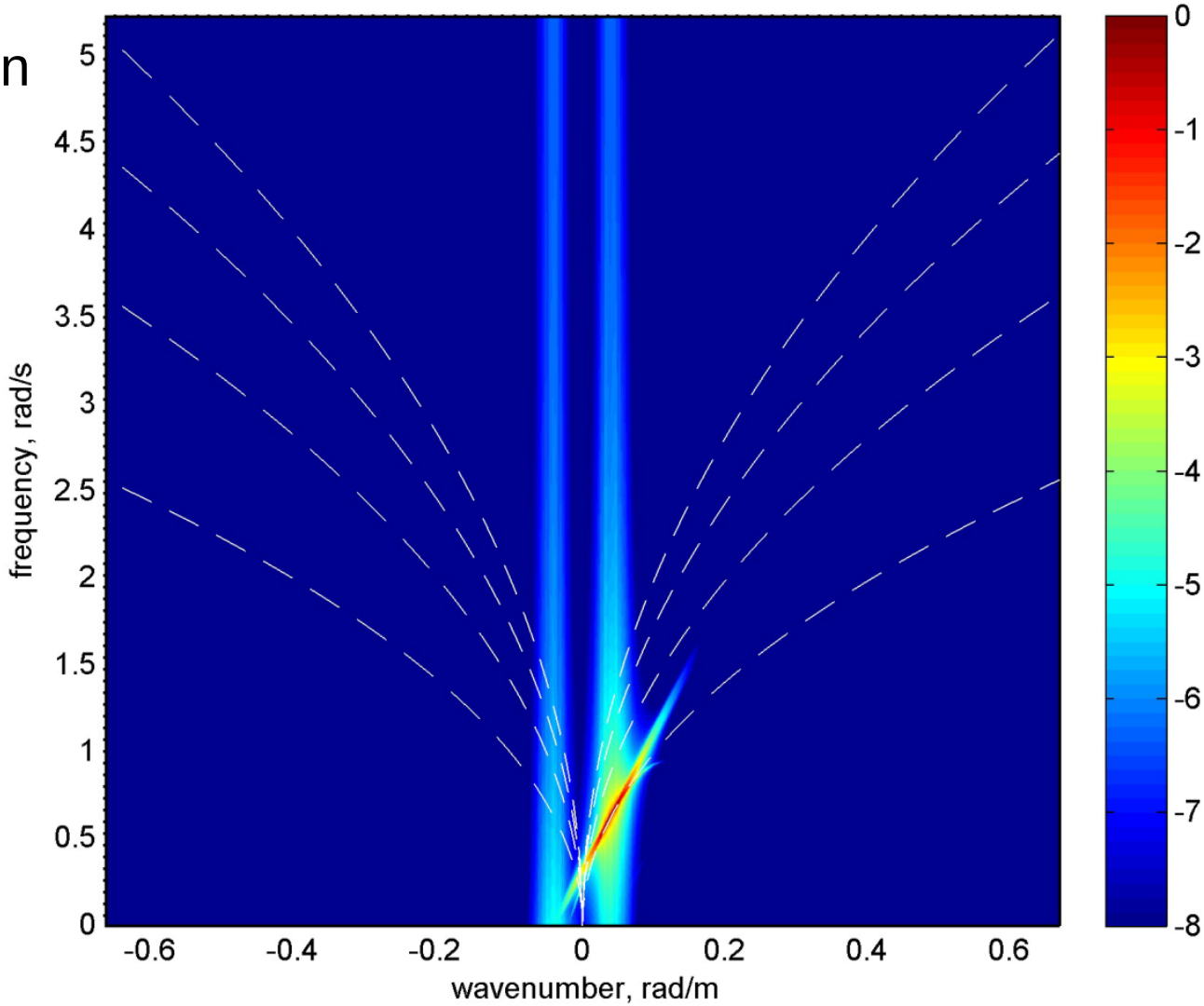
Narrow Gaussian spectrum, nonlinear Schrodinger eq-n

$H_s = 3.3$ m, $T_p = 10$ s, $\Delta\omega/\omega = 0.1$, $\varepsilon_s = 0.07$

Describes the evolution of an envelope of free waves under the assumption of slow modulations of small waves

The spectrum does not follow the dispersive relation

Formation of coherent wave groups – envelope solitons



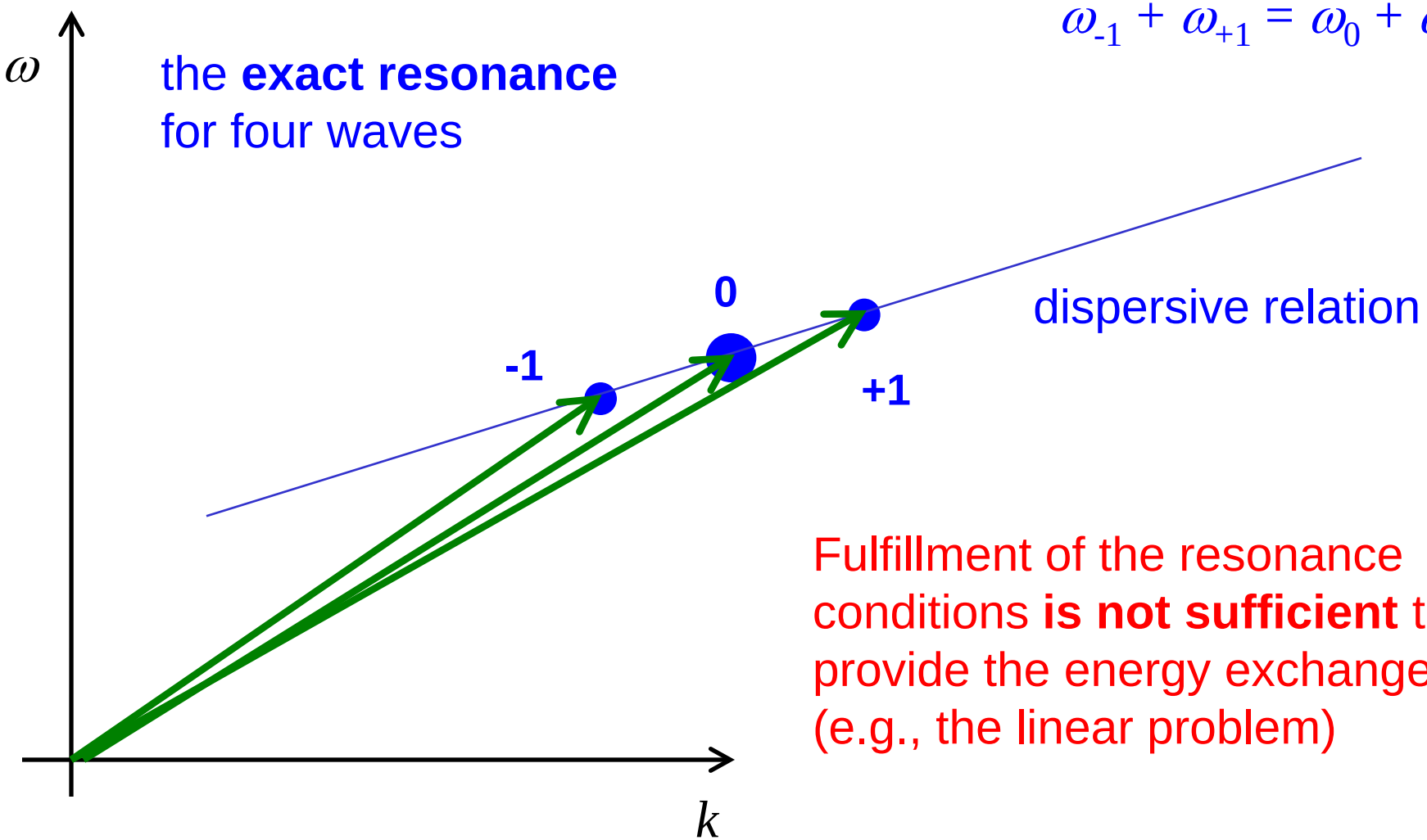
Nonlinear wave interactions

The qualitative description

$$k_{-1} + k_{+1} = k_0 + k_0$$

$$\omega_{-1} + \omega_{+1} = \omega_0 + \omega_0$$

the **exact resonance** for four waves



Fulfillment of the resonance conditions **is not sufficient** to provide the energy exchange (e.g., the linear problem)

Nonlinear wave interactions

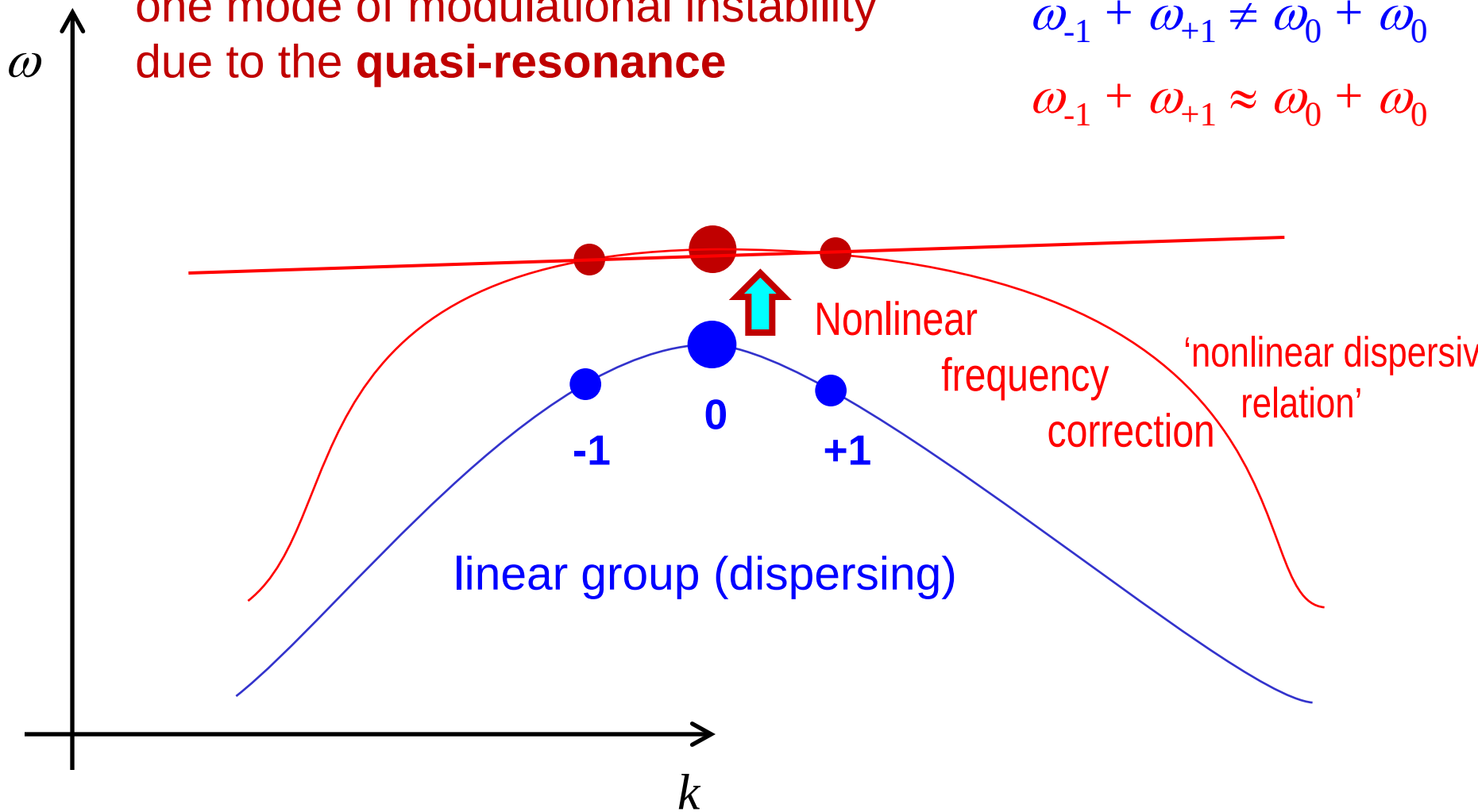
The qualitative description

a **coherent wave quartet** =
one mode of modulational instability
due to the **quasi-resonance**

$$k_{-1} + k_{+1} = k_0 + k_0$$

$$\omega_{-1} + \omega_{+1} \neq \omega_0 + \omega_0$$

$$\omega_{-1} + \omega_{+1} \approx \omega_0 + \omega_0$$



Nonlinear wave interactions

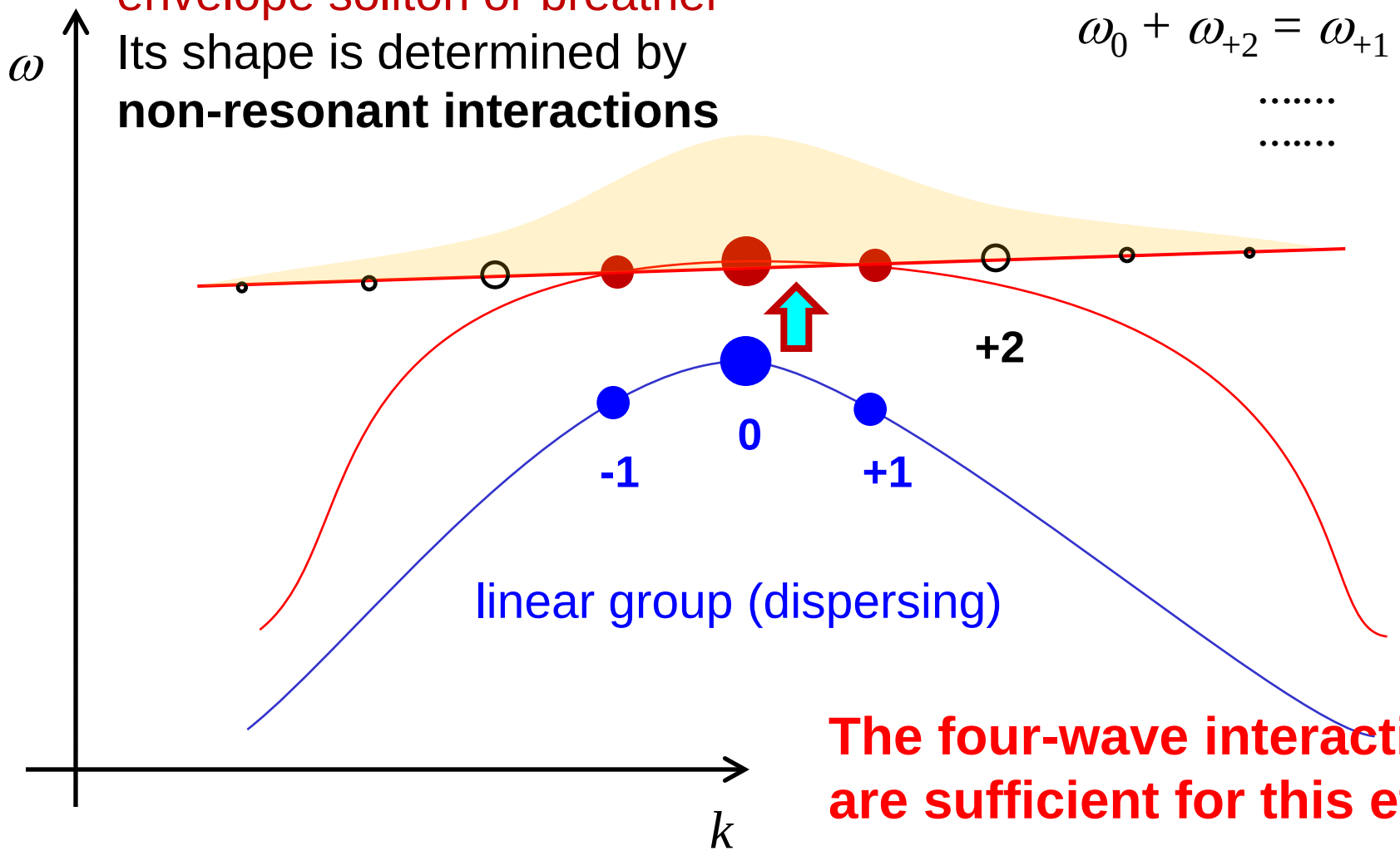
The qualitative description

a coherent wave group =
envelope soliton or breather
Its shape is determined by
non-resonant interactions

$$k_0 + k_{+2} = k_{+1} + k_{+1}$$

$$\omega_0 + \omega_{+2} = \omega_{+1} + \omega_{+1}$$

.....
.....



The four-wave interactions are sufficient for this effect

Nonlinear wave interactions

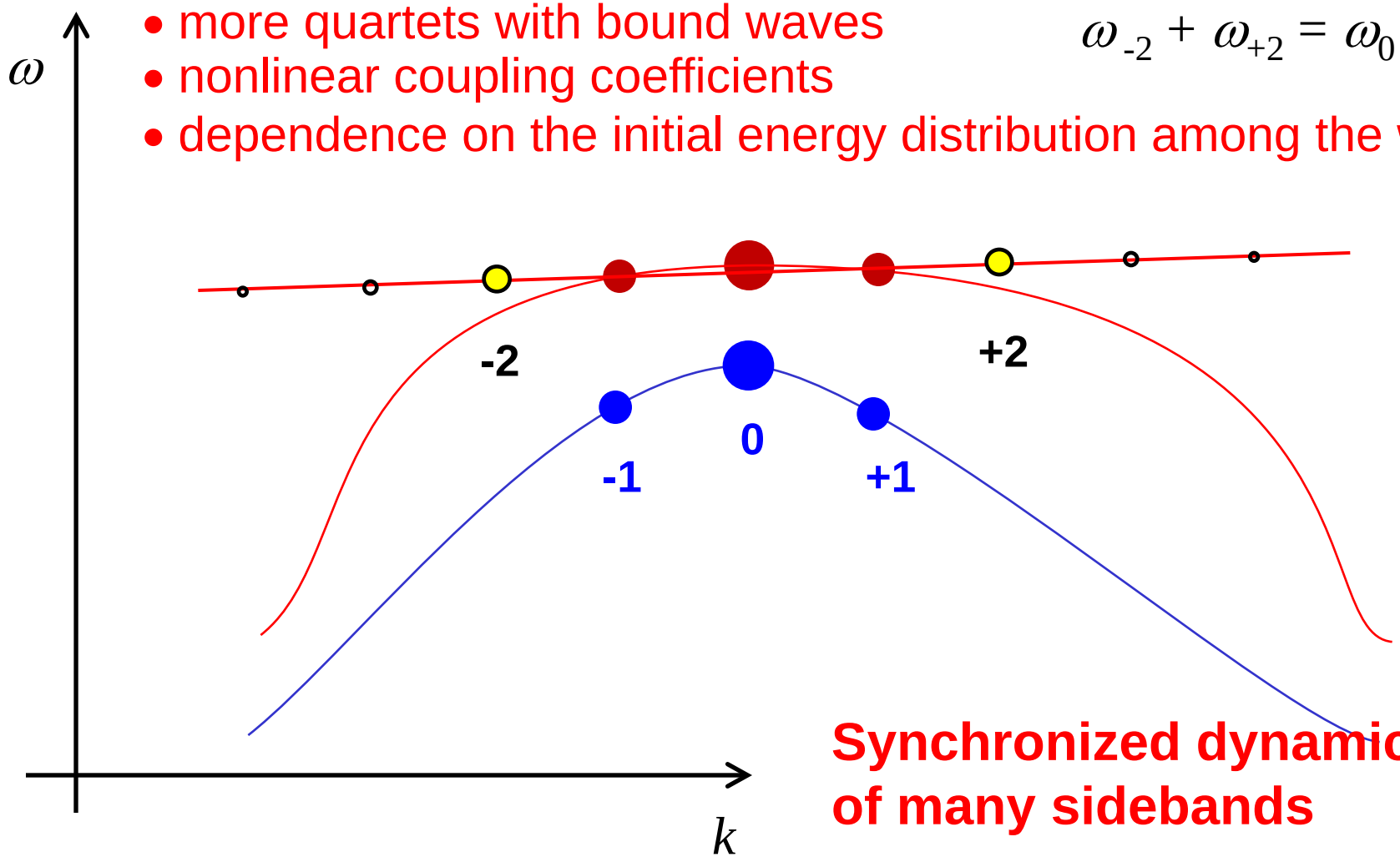
The qualitative description

$$k_{-2} + k_{+2} = k_0 + k_0$$

$$\omega_{-2} + \omega_{+2} = \omega_0 + \omega_0$$

The life is more complicated:

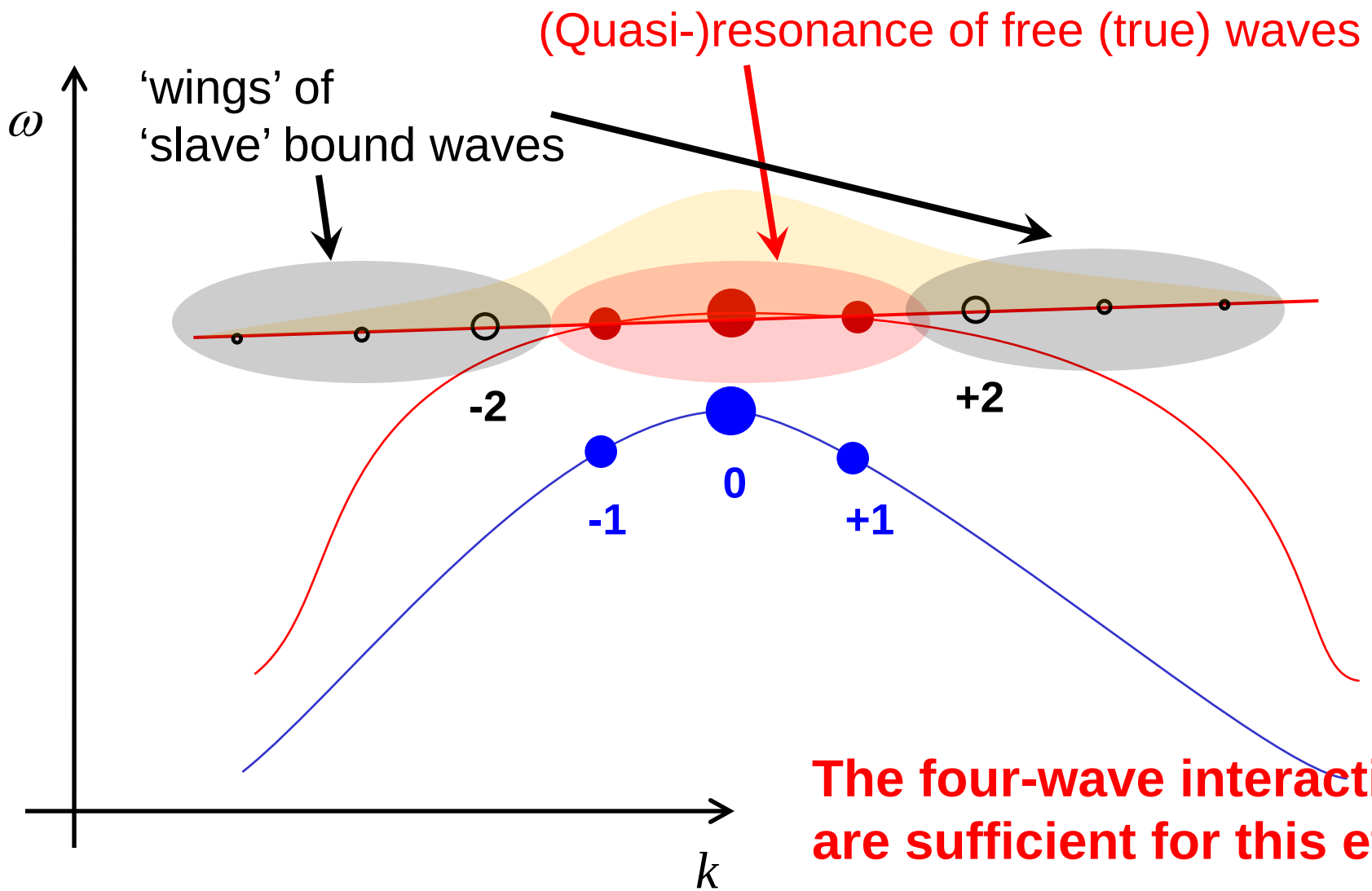
- more quartets with bound waves
- nonlinear coupling coefficients
- dependence on the initial energy distribution among the waves



Synchronized dynamics of many sidebands

Nonlinear wave interactions

The qualitative description



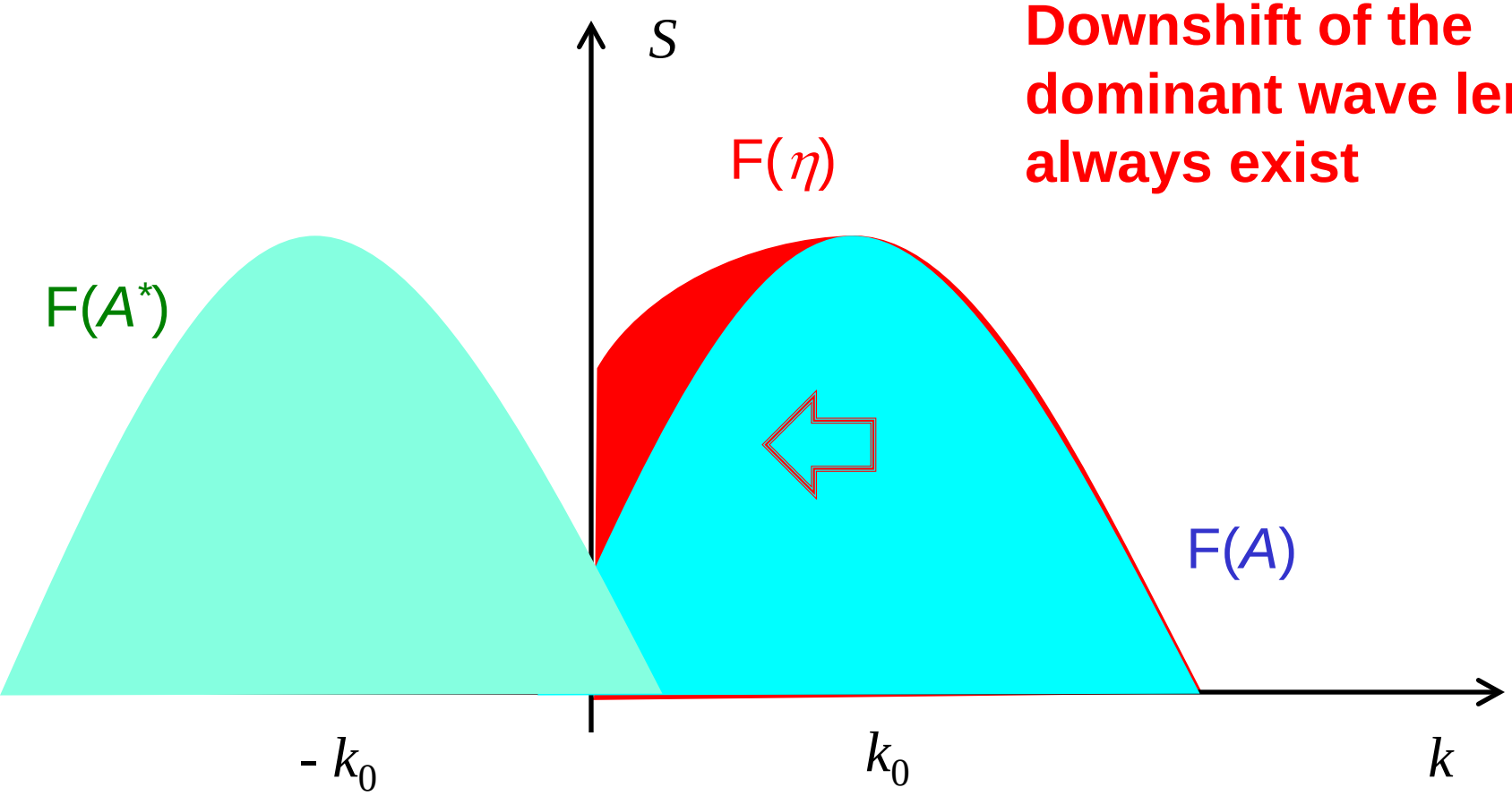
The four-wave interactions are sufficient for this effect

One amusing observation

The very general situation

Modulated signal $\eta(x)$
with complex envelope $A(x)$

$$\eta(x) = A(x)\exp(ik_0 x) + A^*(x)\exp(-ik_0 x)$$



Downshift of the dominant wave length always exist

One amusing observation

The very general situation

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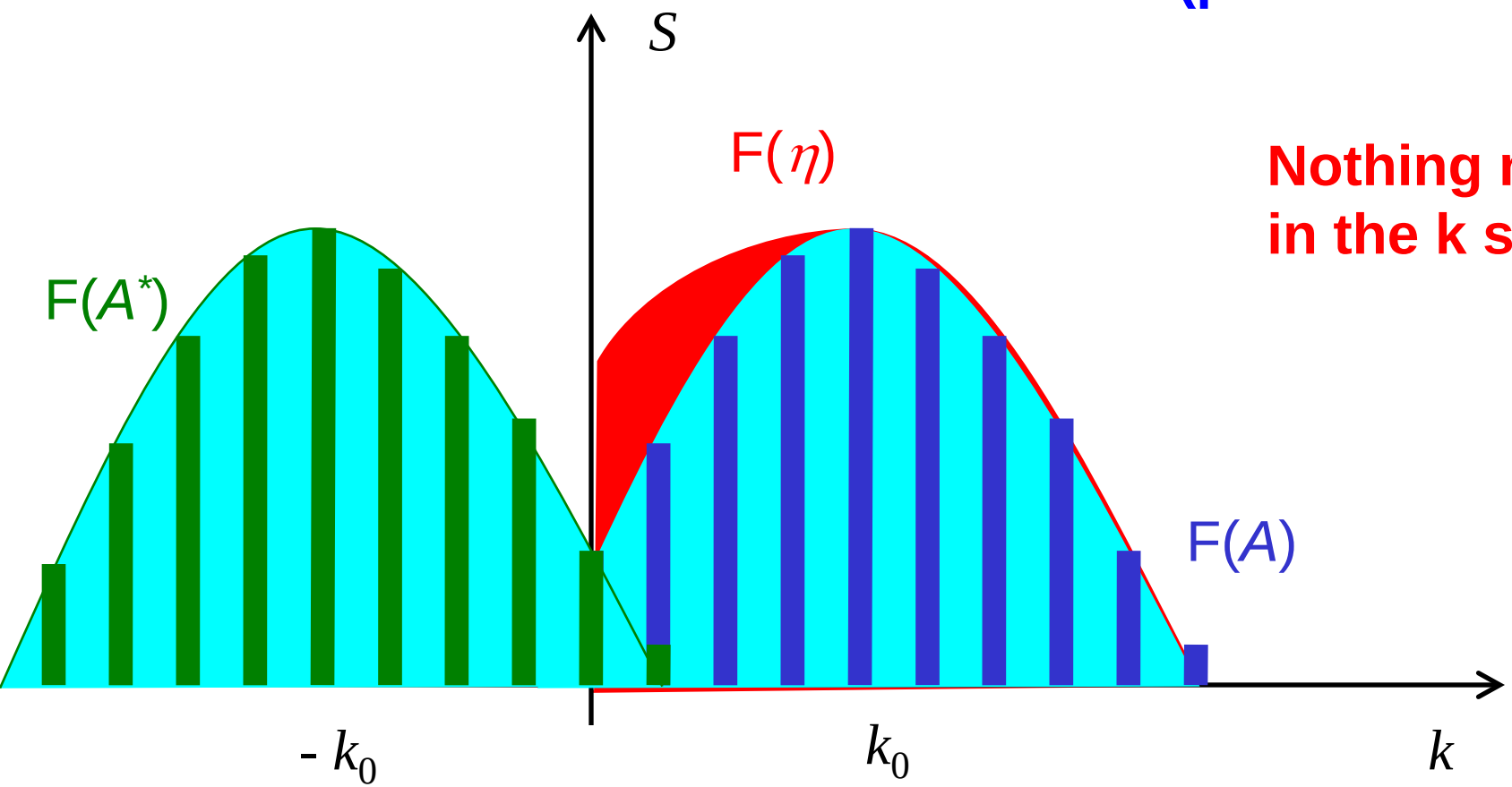
What if discrete *k* (periodic domain)?



Nothing new in the k space

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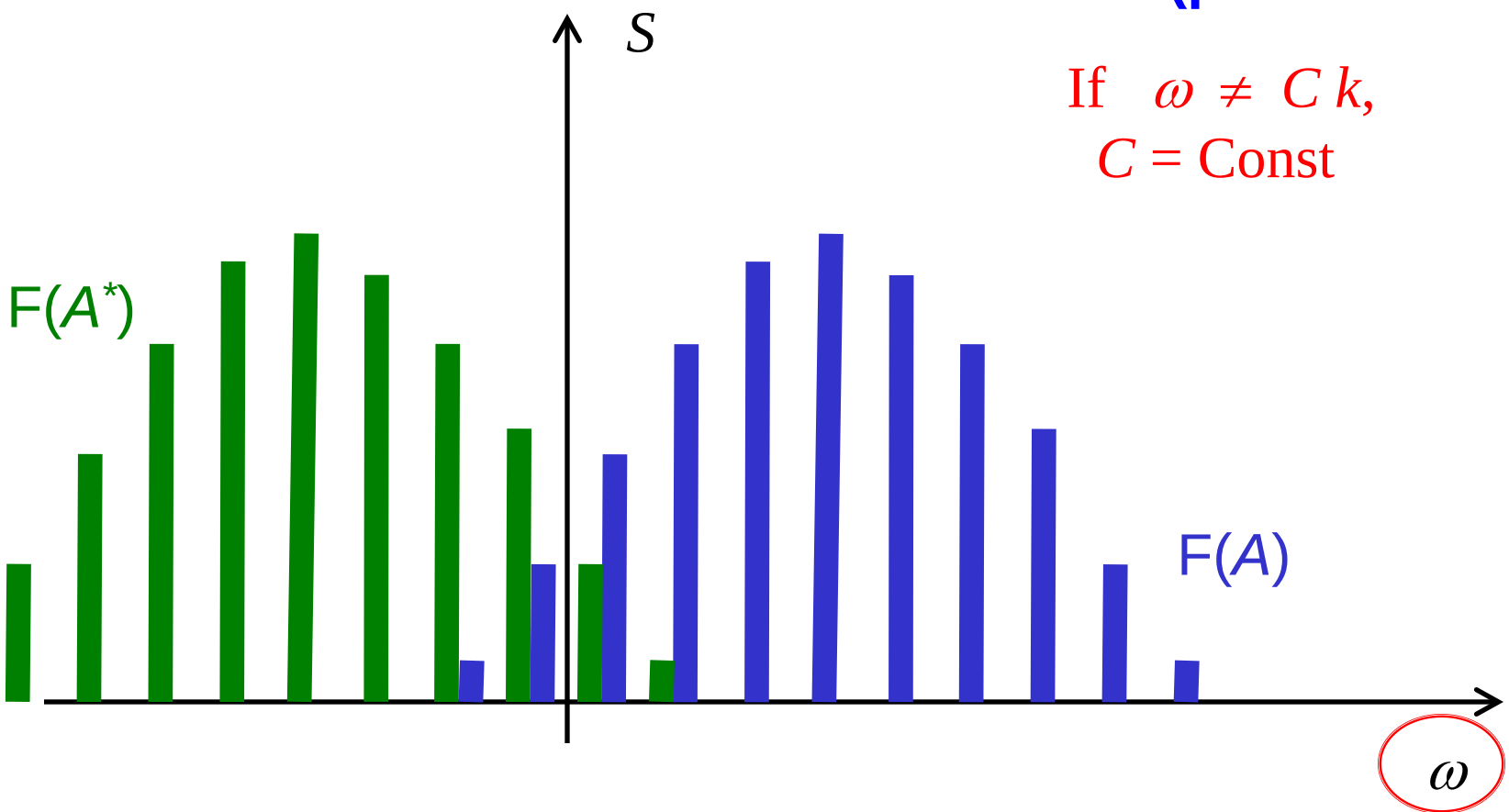
What if discrete *k* (periodic domain)?

If $\omega \neq C\,k$,
$C = \text{Const}$



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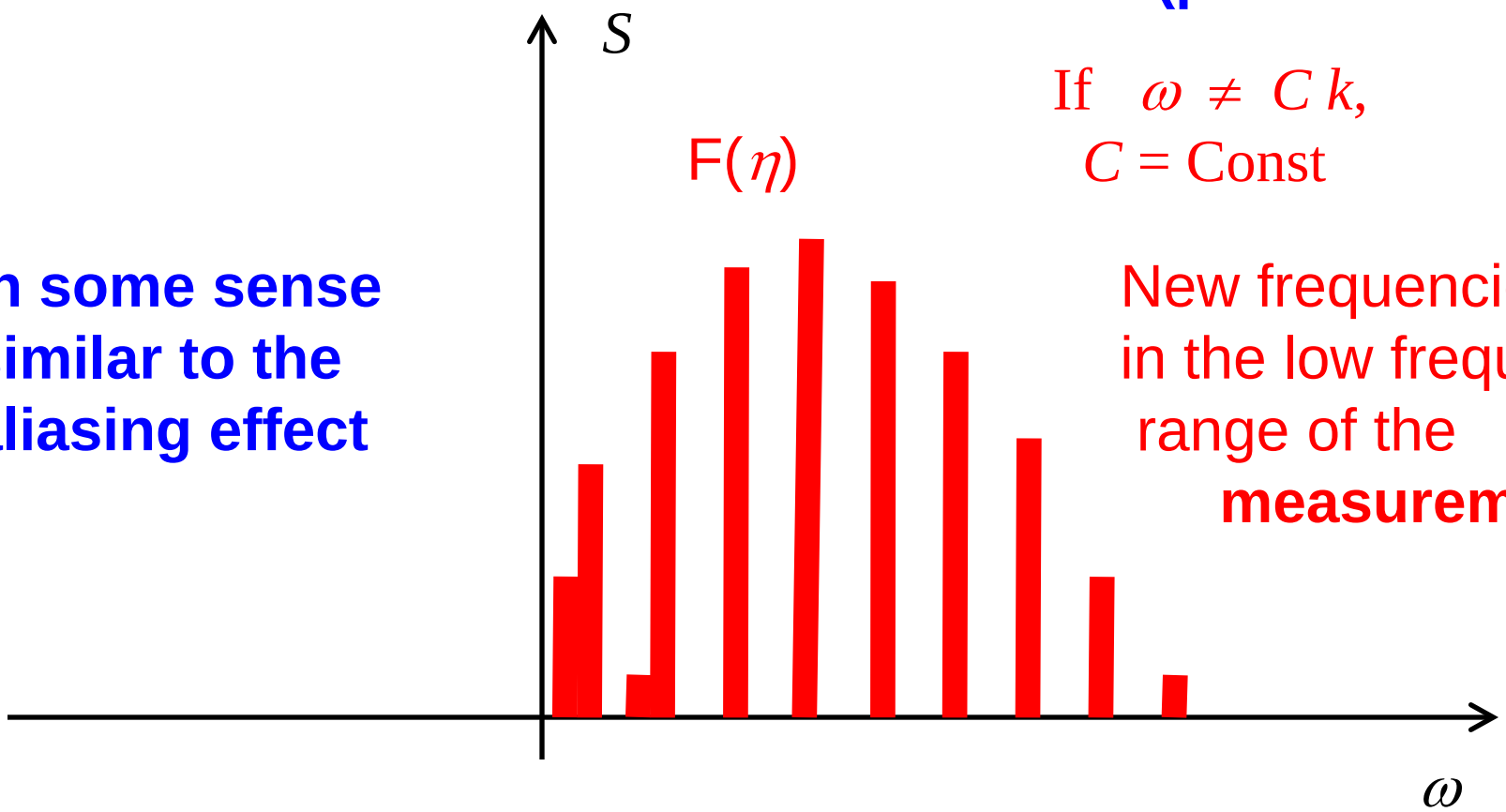
If $\omega \neq C\,k$,
$C = \text{Const}$

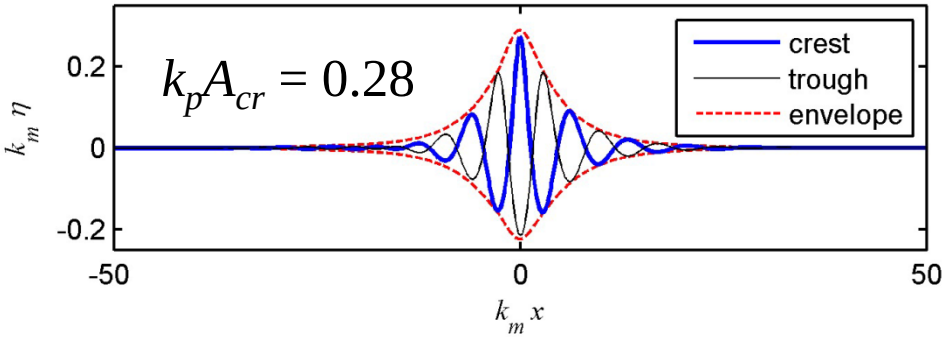
In some sense similar to the aliasing effect

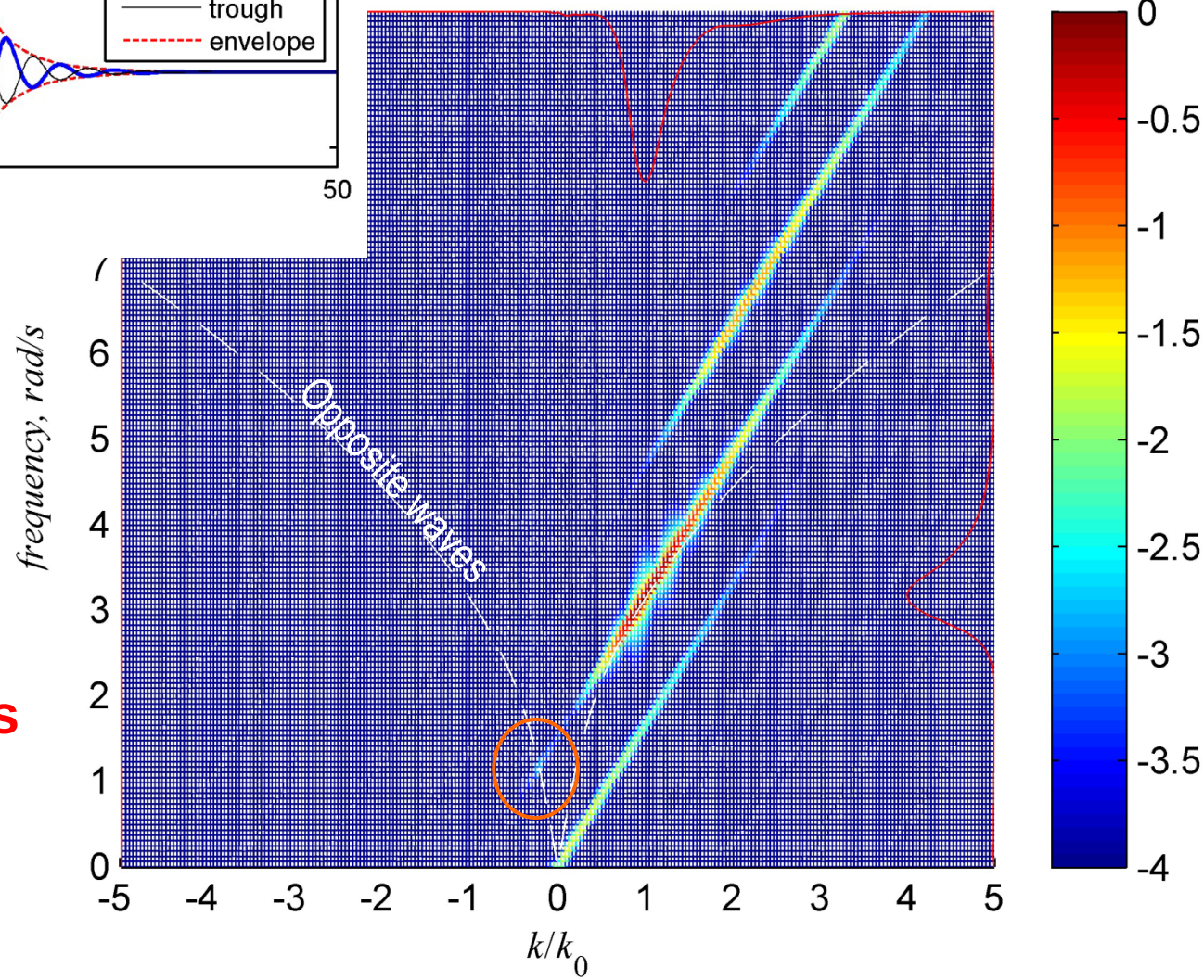


New frequencies in the low frequency range of the **measurement**

Fourier transform of a solitary group

Soliton-like group, primitive potential hydrodyn. eqs.



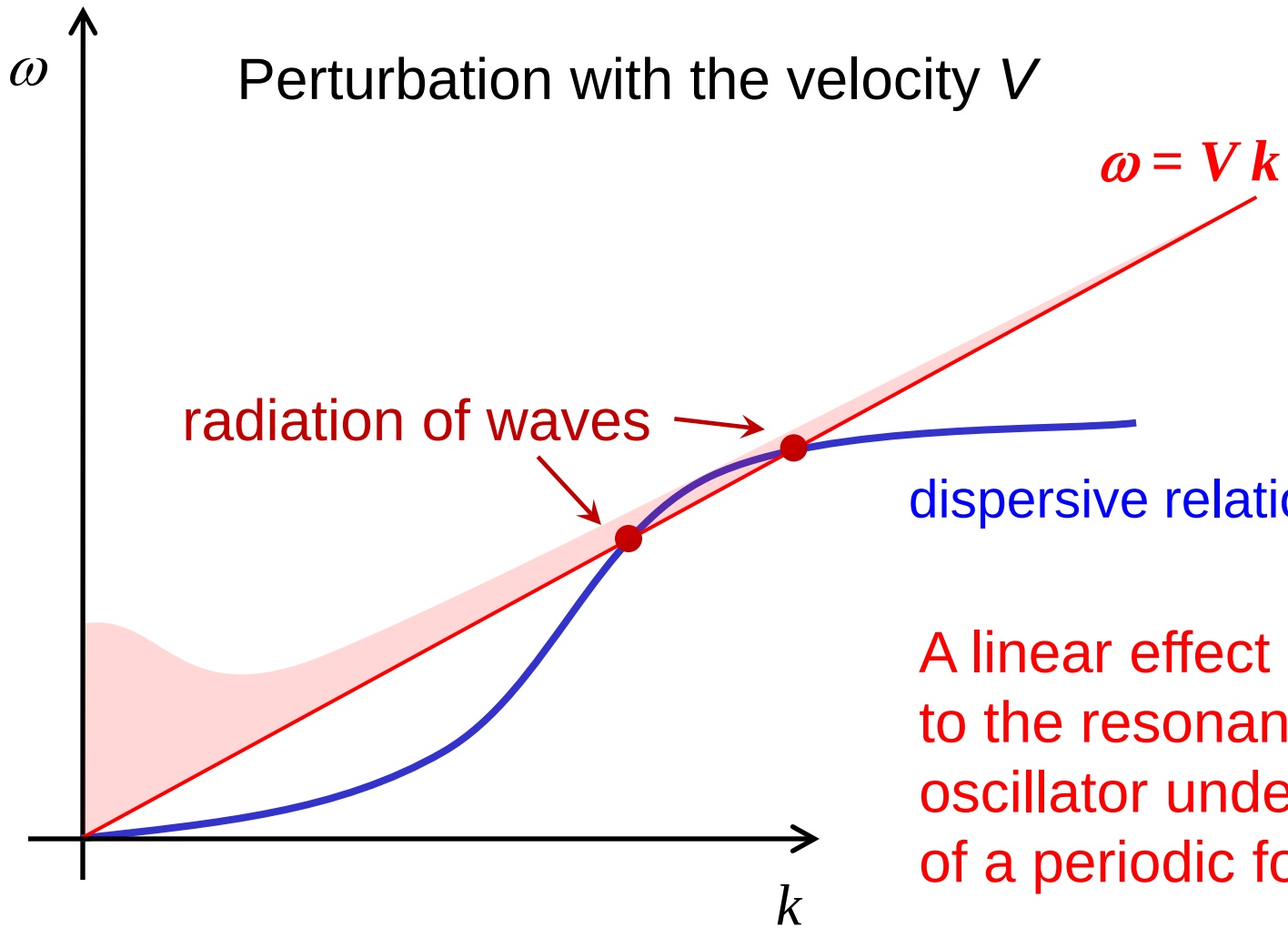


Effects of nonlinearity are not limited by the frequency shift!

Intense wave groups may result in generation of other waves?

Resonance between waves and moving objects (Cherenkov radiation)

The general idea

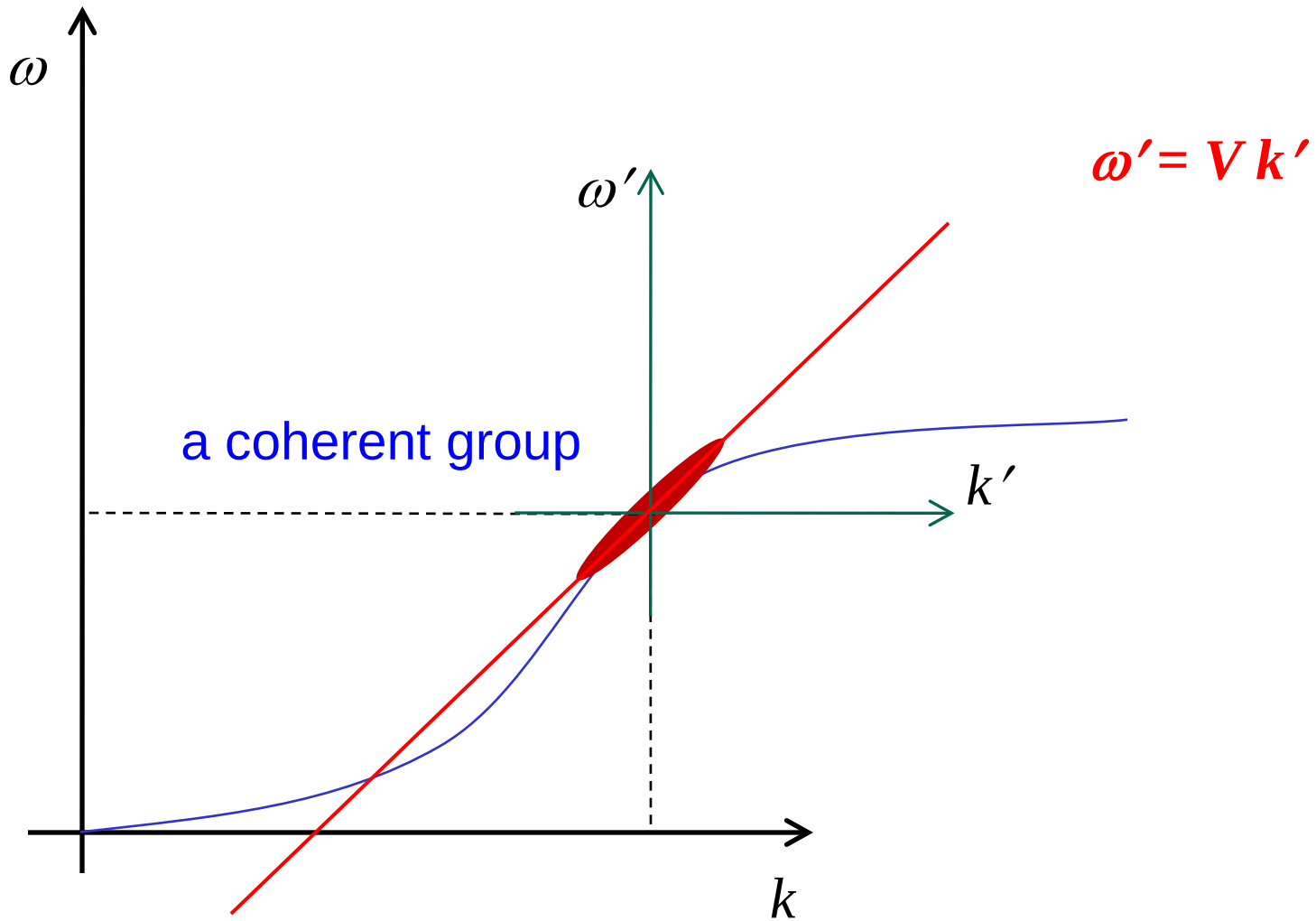


$$V = \frac{\Omega}{k} = C_{ph}$$

A linear effect similar to the resonance of an oscillator under the action of a periodic force

Resonance between waves and groups

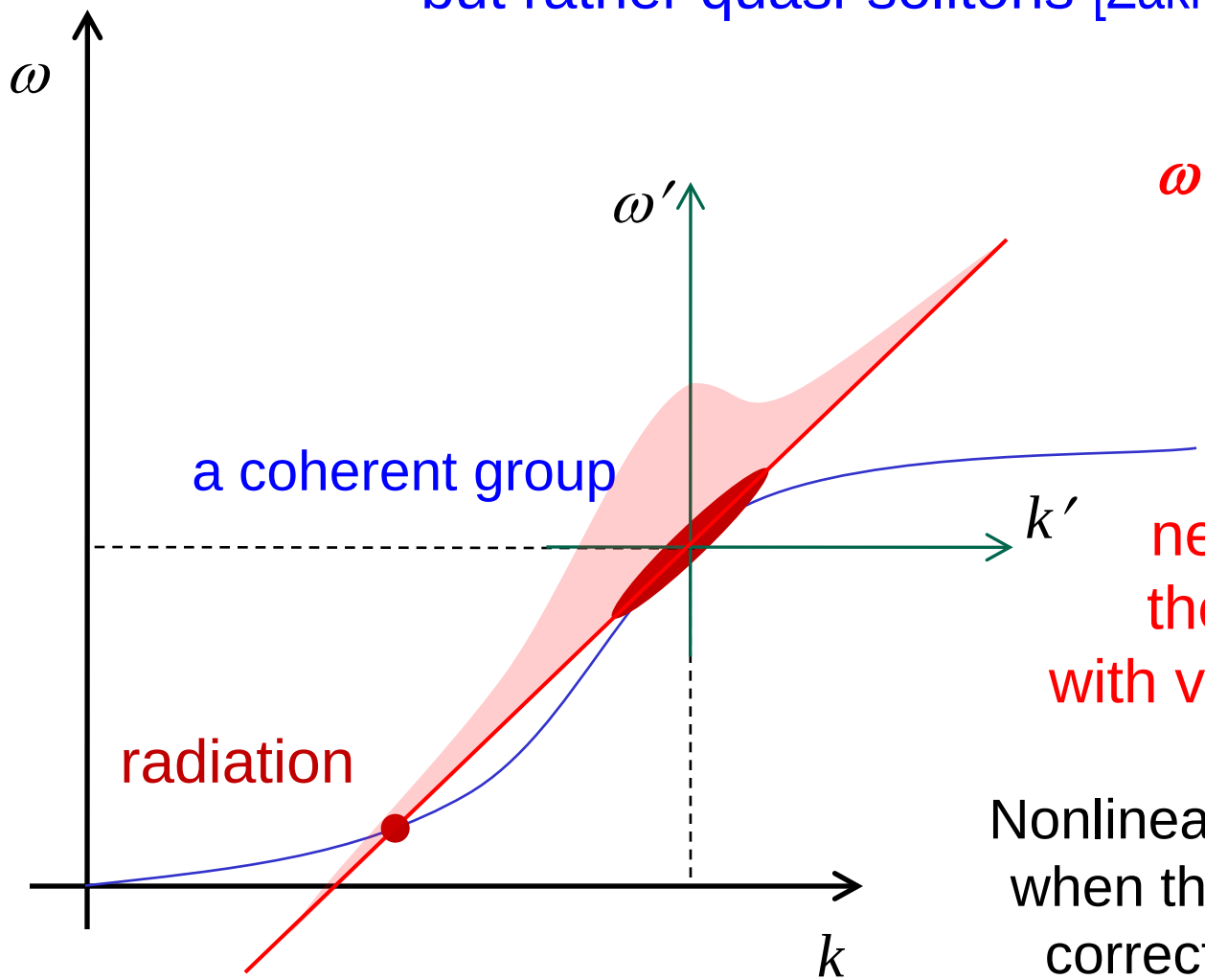
The general idea



Resonance between waves and groups

The general idea

If such interaction may occur, the groups are not solitons, but rather quasi-solitons [Zakharov & Kuznetsov, 1998]



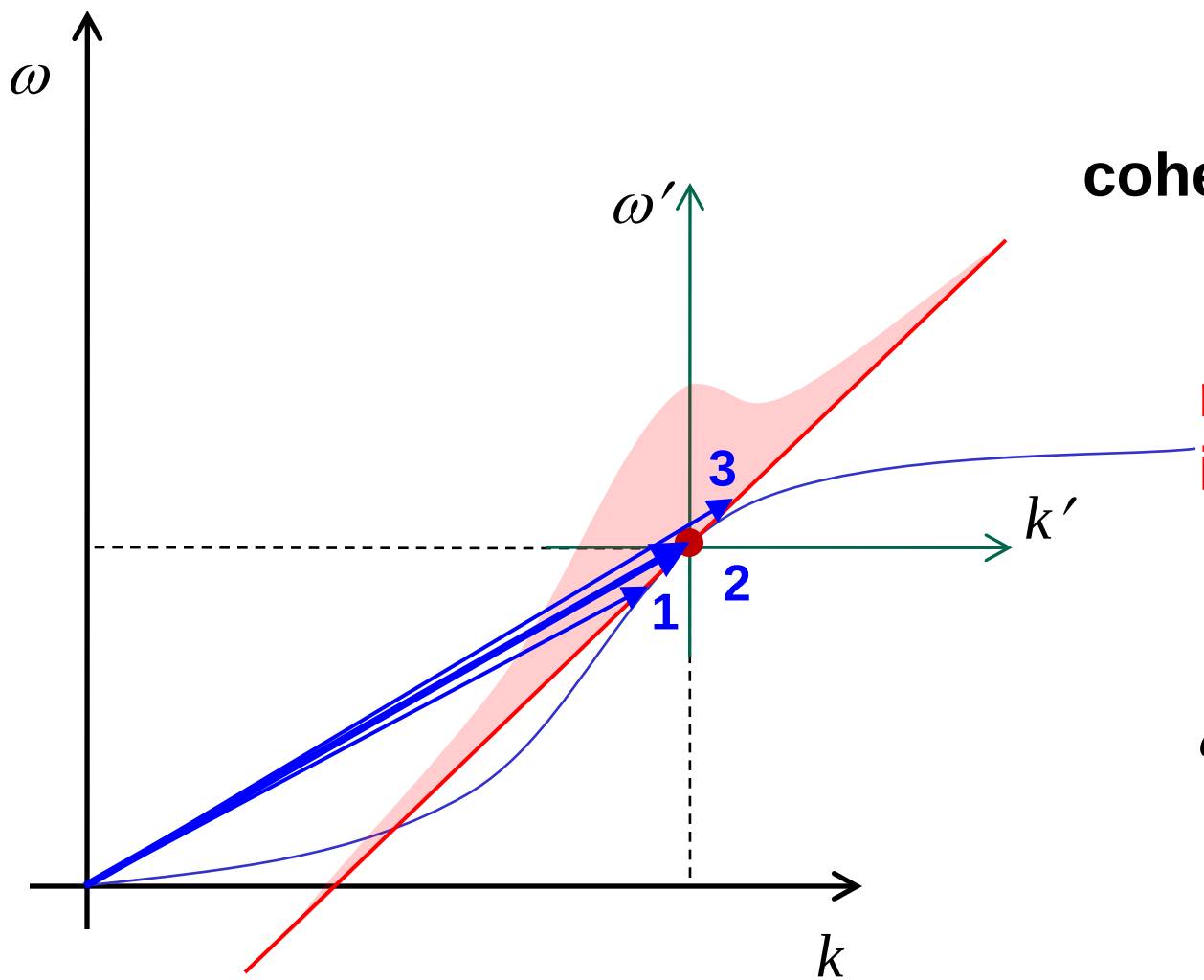
$\omega' = V k'$

Nonlinearity is necessary to maintain the group propagating with velocity V as a whole.

Nonlinear Cherenkov radiation when the nonlinear frequency correction taken into account

Resonance between waves and groups

The general idea



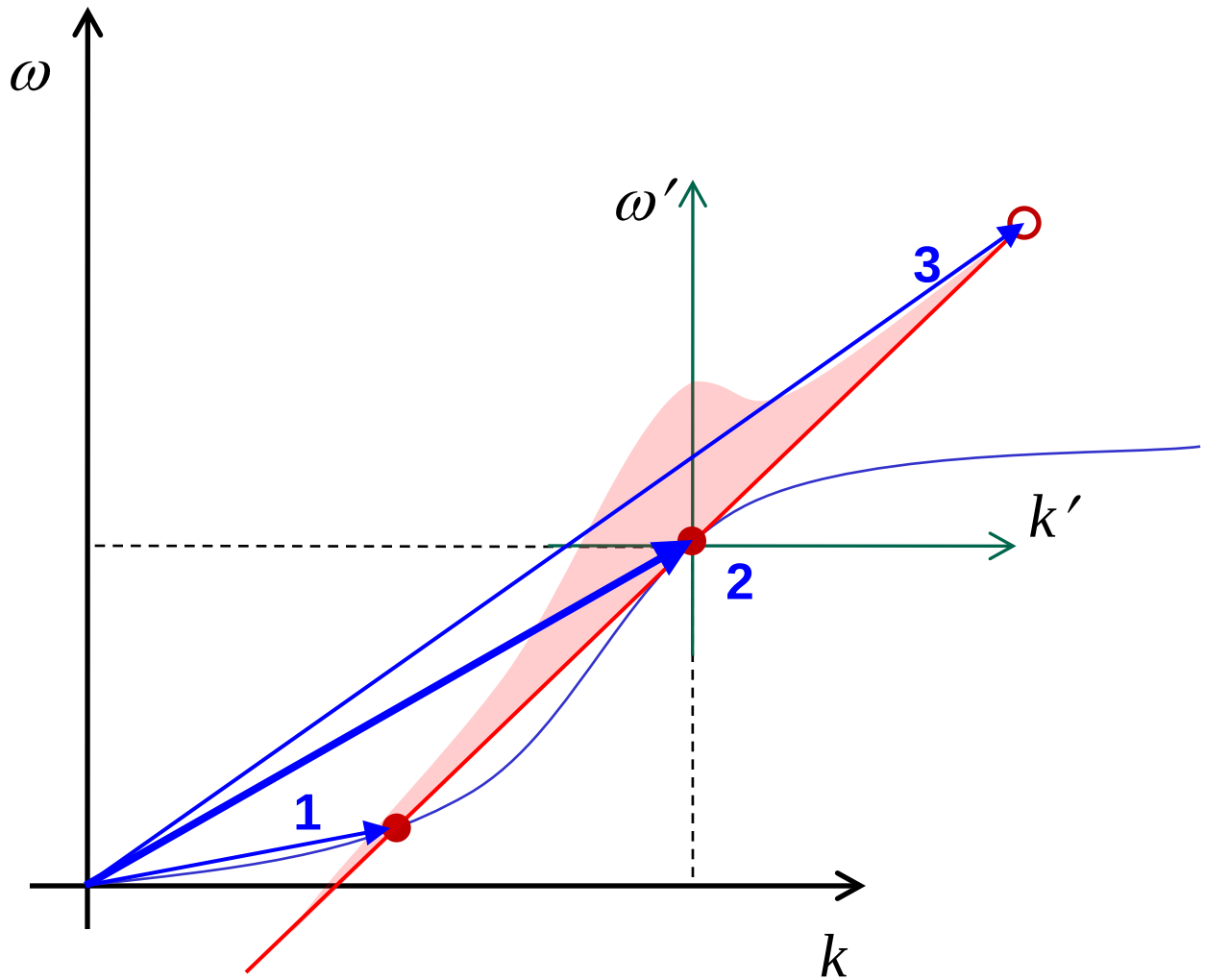
Formation of the coherent nolinear wave group is a quasi-resonant & non-resonant four-wave interaction process

$$k_1 + k_3 = k_2 + k_2$$

$$\omega_1 + \omega_3 = \omega_2 + \omega_2$$

Resonance between waves and groups

The general idea



Then, the wave emission it is a strongly non-resonant four-wave interaction

$$k_1 + k_3 = k_2 + k_2$$

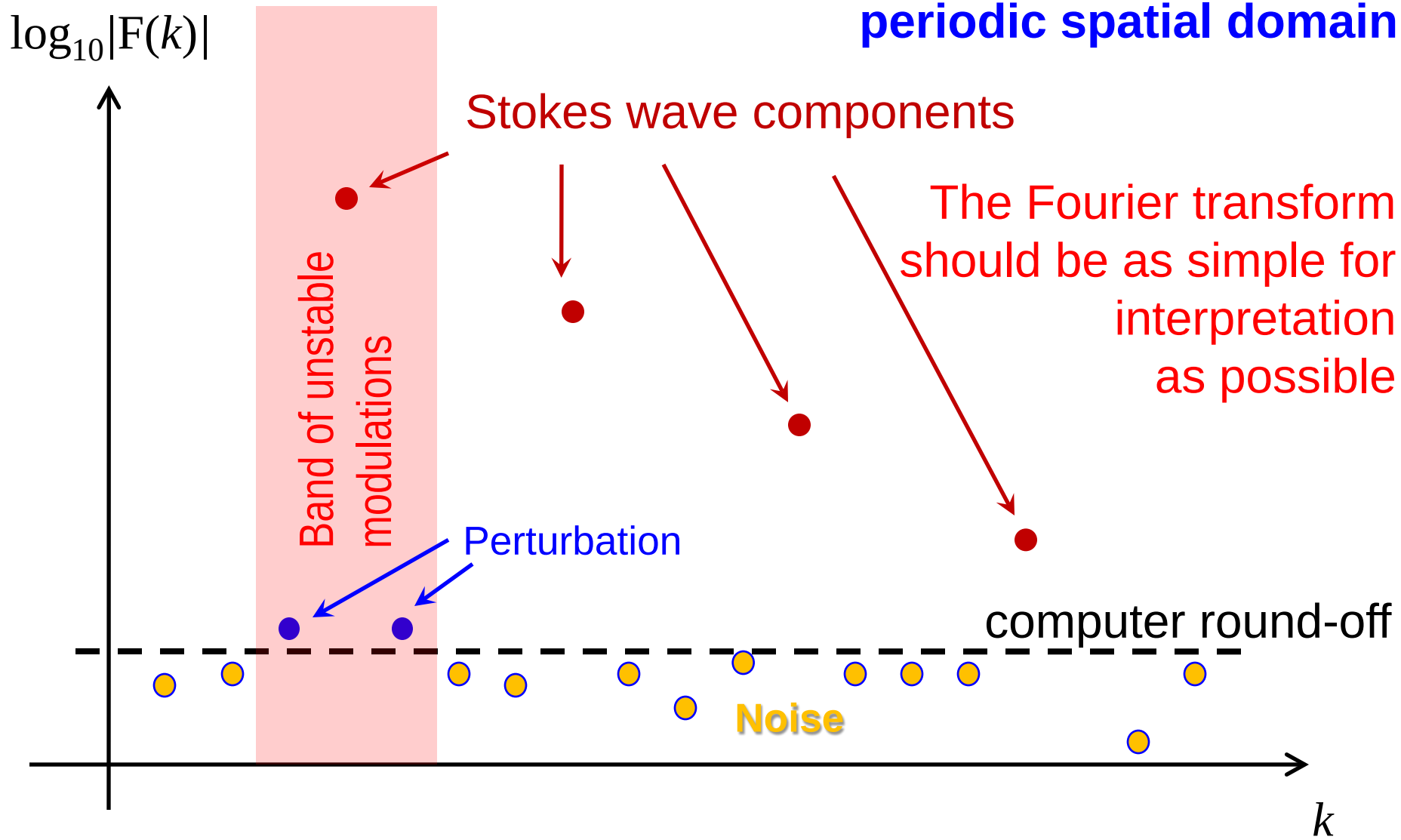
$$\omega_1 + \omega_3 = \omega_2 + \omega_2$$

4. Nonlinear wave-group resonances

[Accurate numerical simulations of the modulational instability]

Nonlinear modulated waves

The initial condition: one nonlinear wave



Nonlinear modulated waves

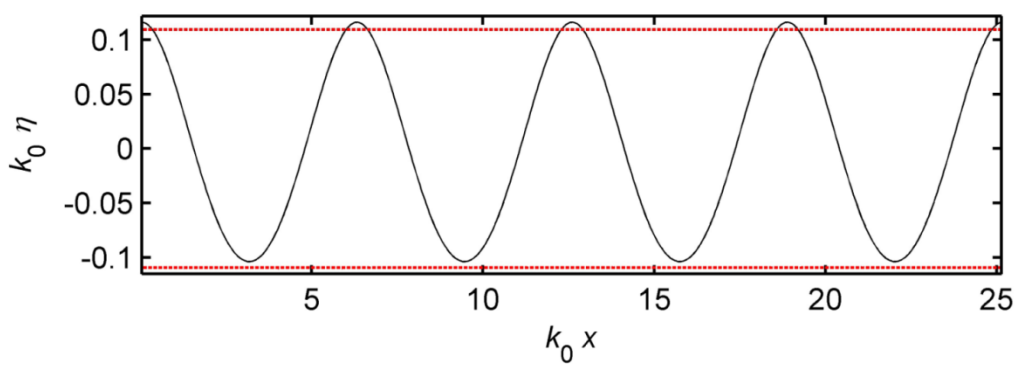
The initial condition

The ‘numerically exact’ Stokes wave, i.e., the stationary solution of the Euler eqs. for the potential movement of ideal fluid.
Due to the choice of the wave steepness, $\varepsilon \equiv k_0 H/2$ and the size of the periodic computational domain, L,
only one mode of the modulational instability may develop.

The initial perturbations are at the level of the computer round-off.

Simulation of the full potential hydrodynamic eqs in conformal variables.

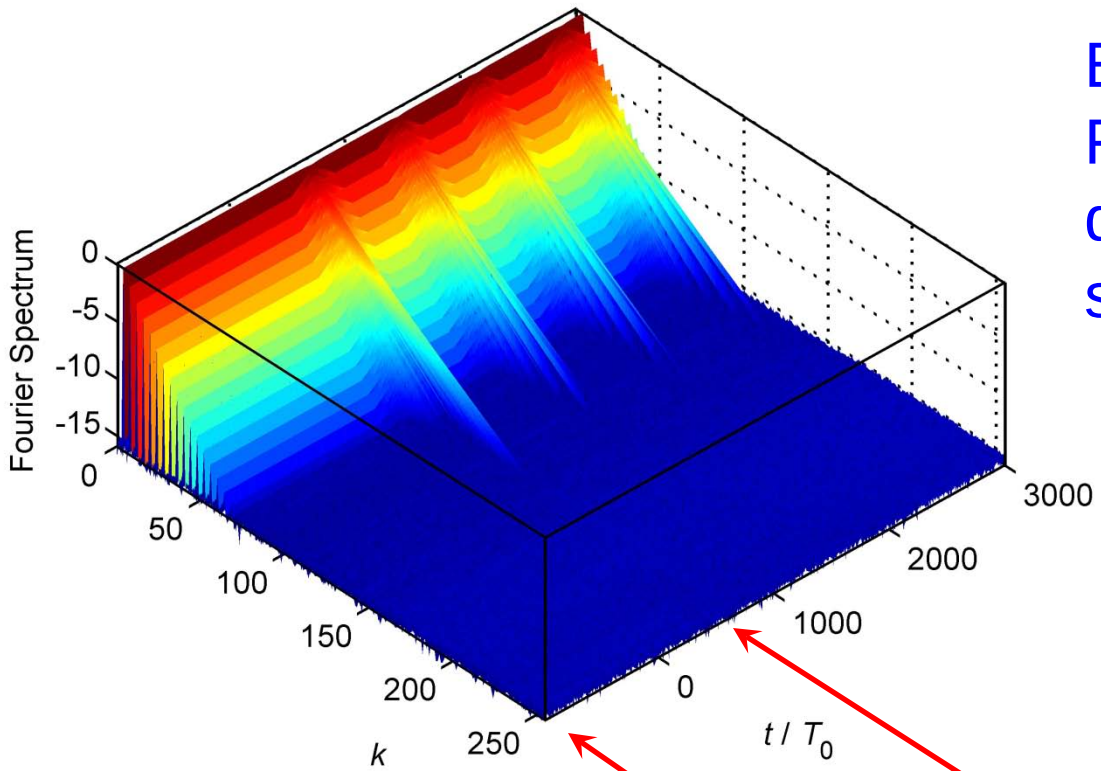
The Fourier transform should be as simple for interpretation as possible



$\varepsilon = 0.11$, $L/\lambda = 4$

Nonlinear modulated waves

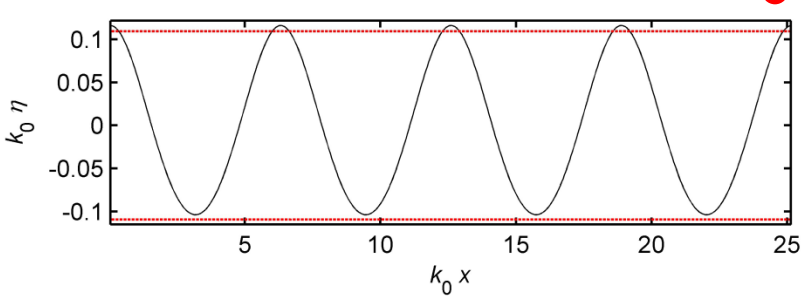
The modulational instability

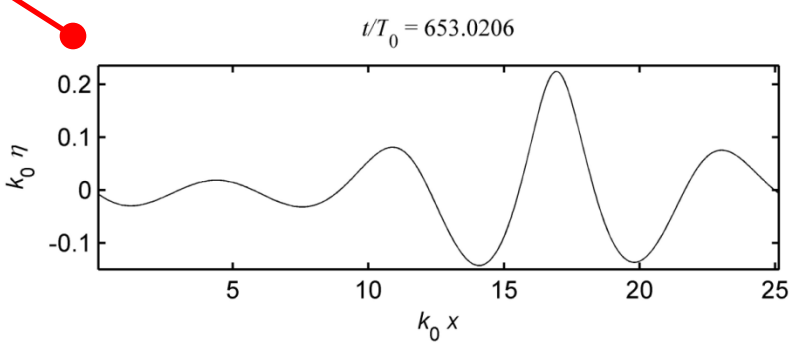


Evolution of the spatial Fourier transform: quasi-periodic self-modulation circles

Breather-type solution of the primitive equations of hydrodynamics

The maximum wave





Nonlinear modulated waves

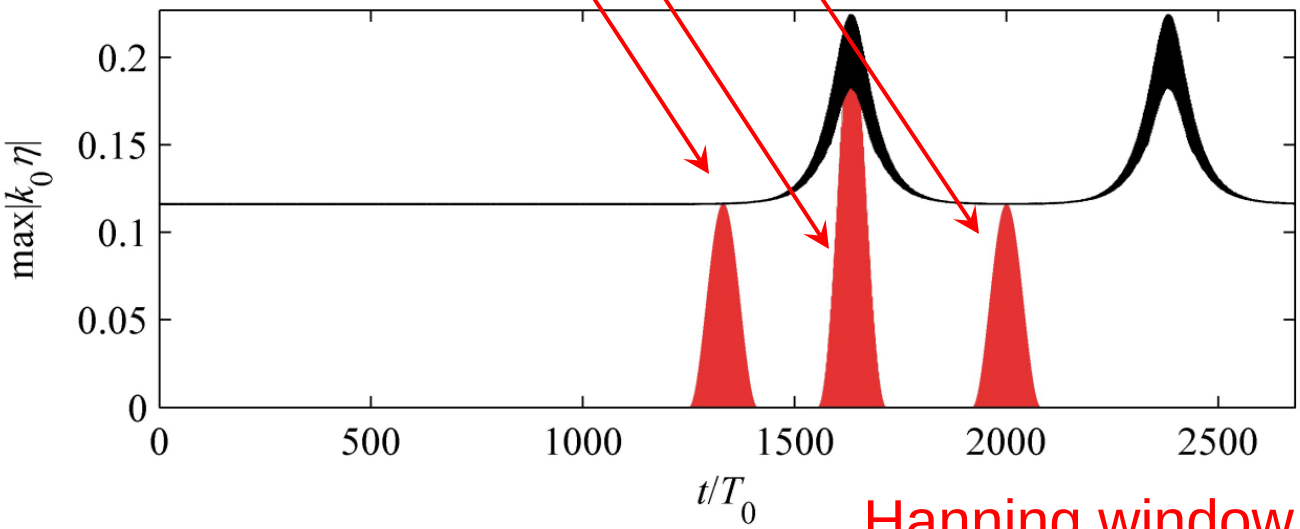
The modulational instability

Evolution of the maximum surface elevation and three selections:

i) Before the wave focusing

ii) At the moment of the focusing event

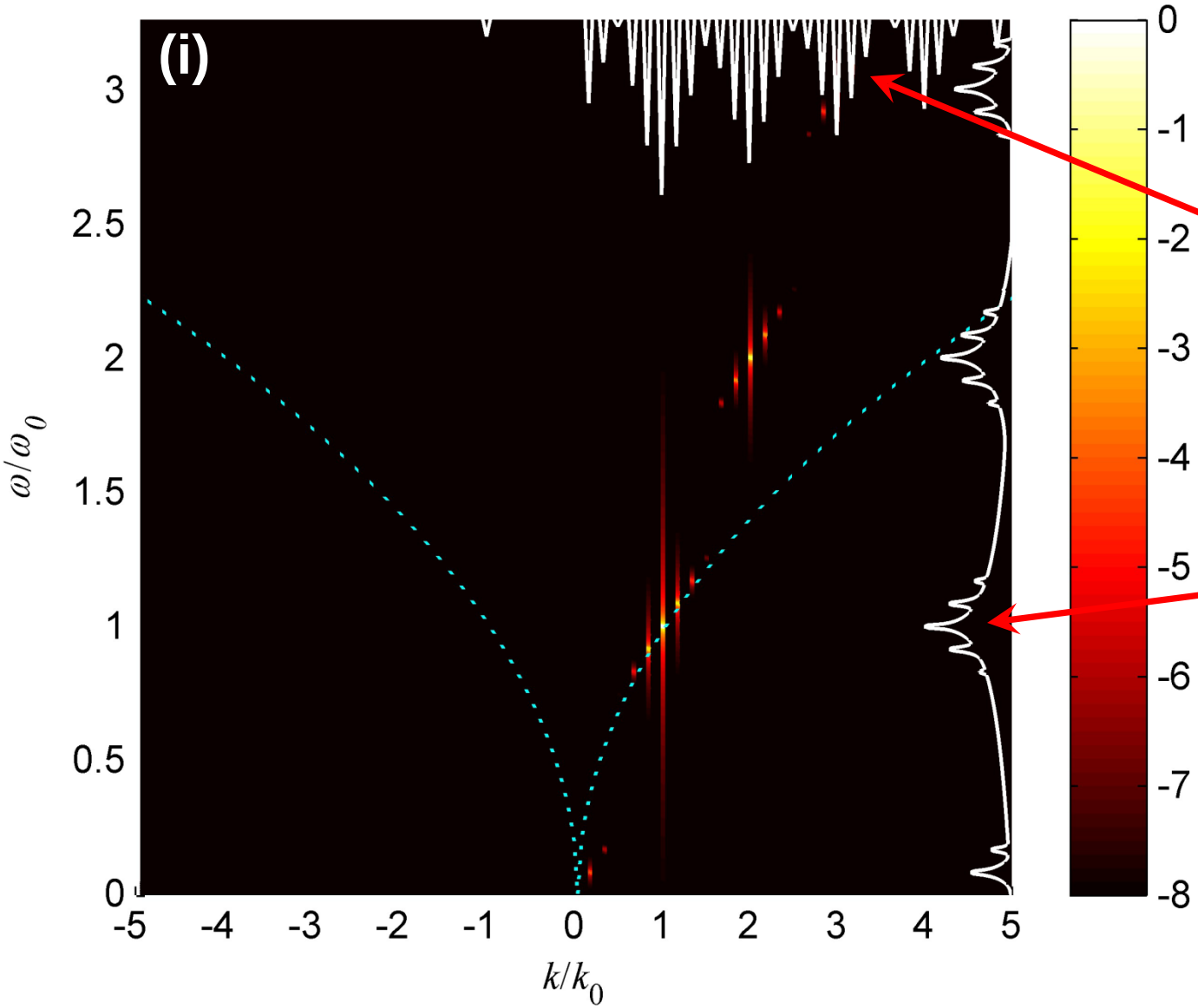
iii) Between two consecutive focuses



Hanning window in time

Double Fourier transform plane

The modulational instability: growing modulations



$\varepsilon = 0.07$, $L/\lambda = 6$

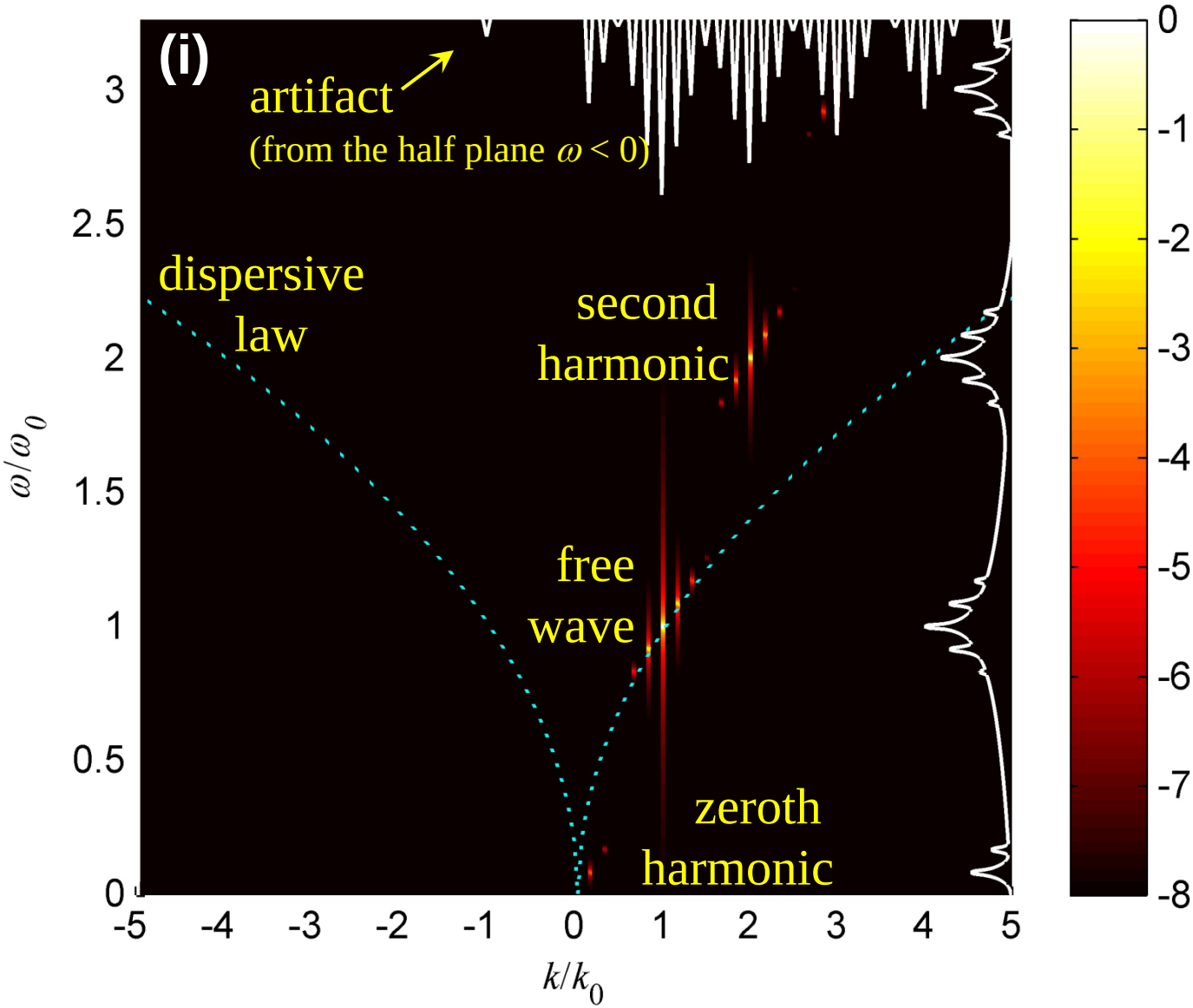
Wavenumber Fourier transform above

Frequency Fourier transforms at the sides

Normalized Fourier amplitudes are shown in the logarithmic scale, $\log_{10}F(\eta(x,t))$

Double Fourier transform plane

The modulational instability: growing modulations



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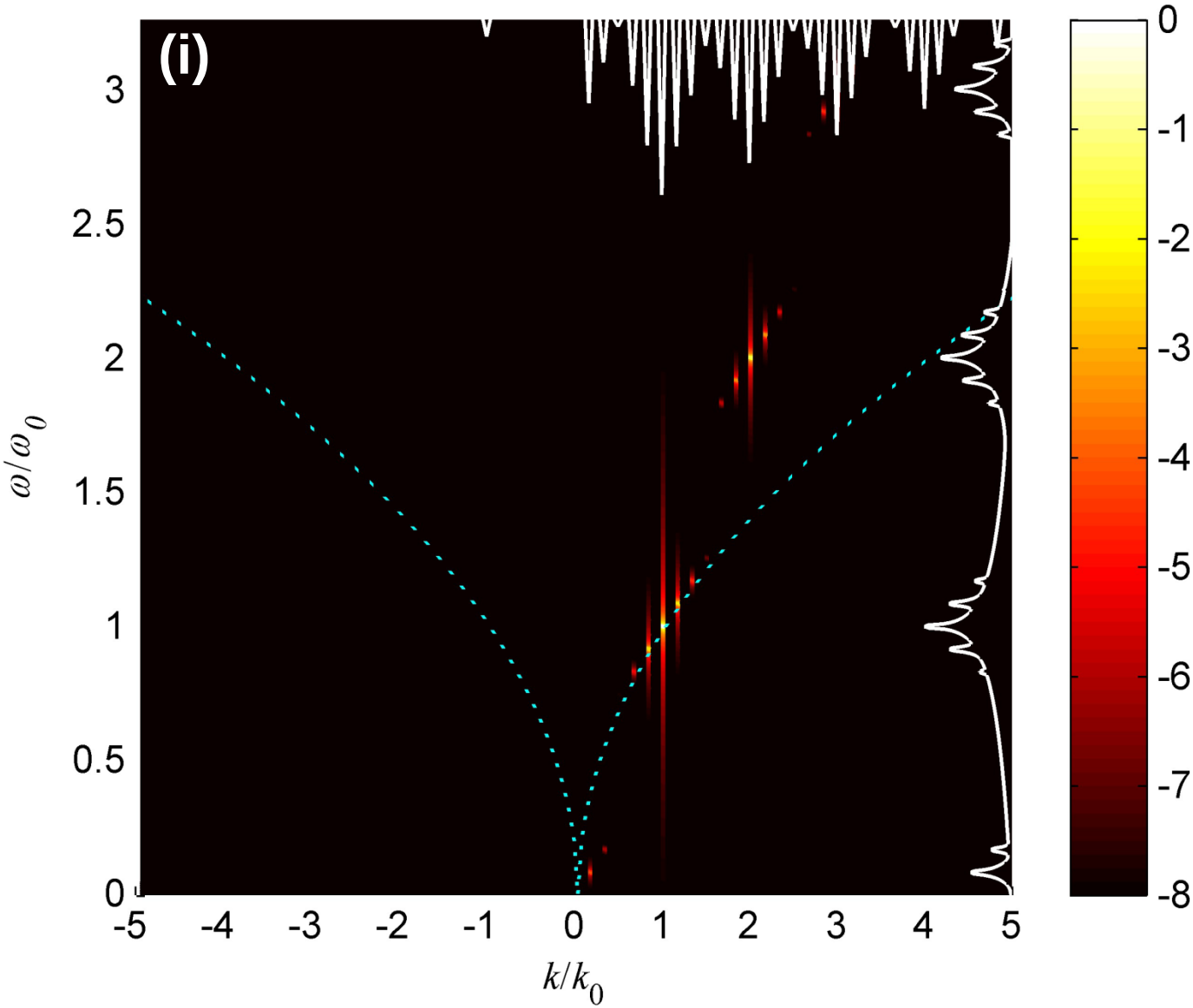

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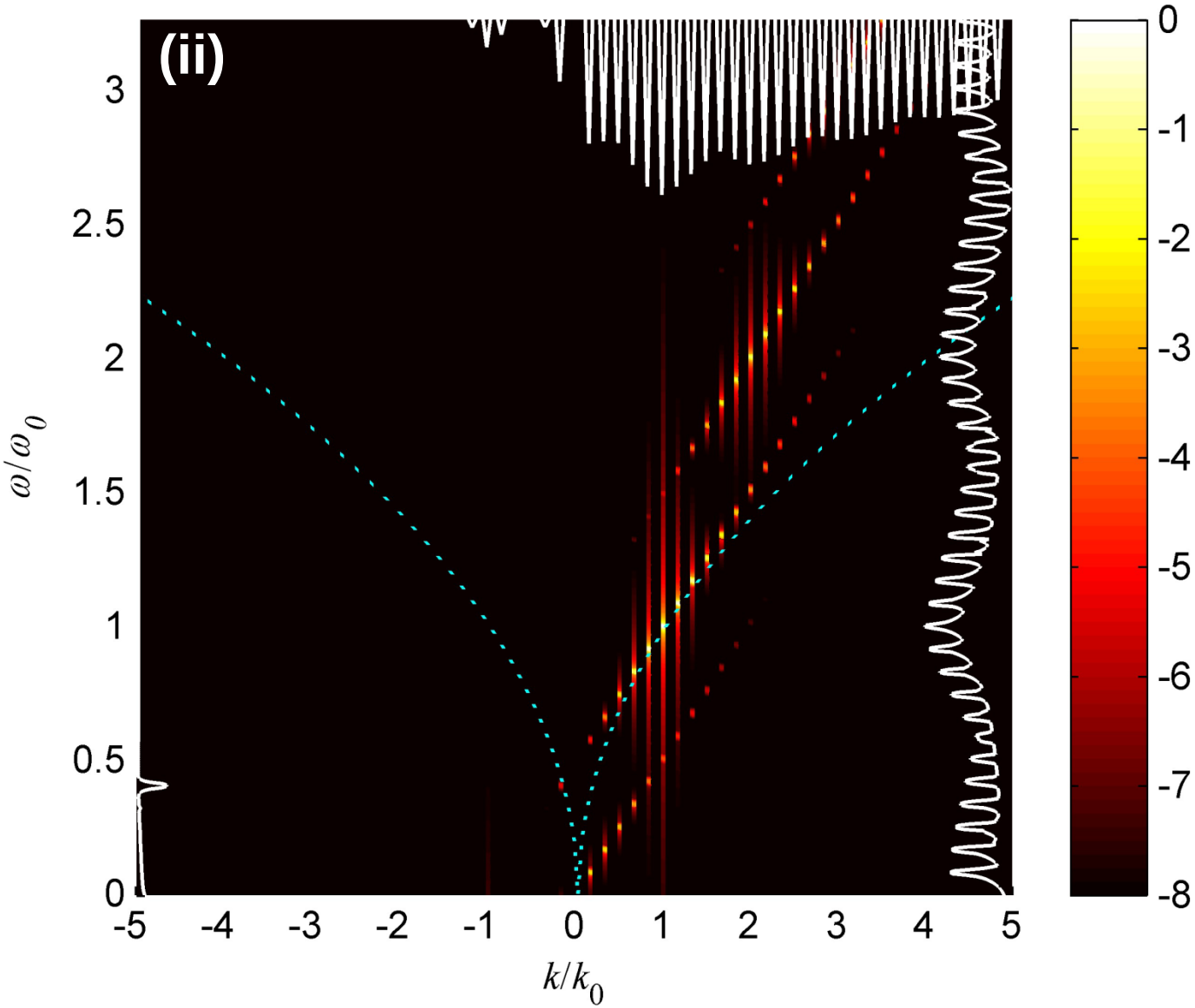
Double Fourier transform plane

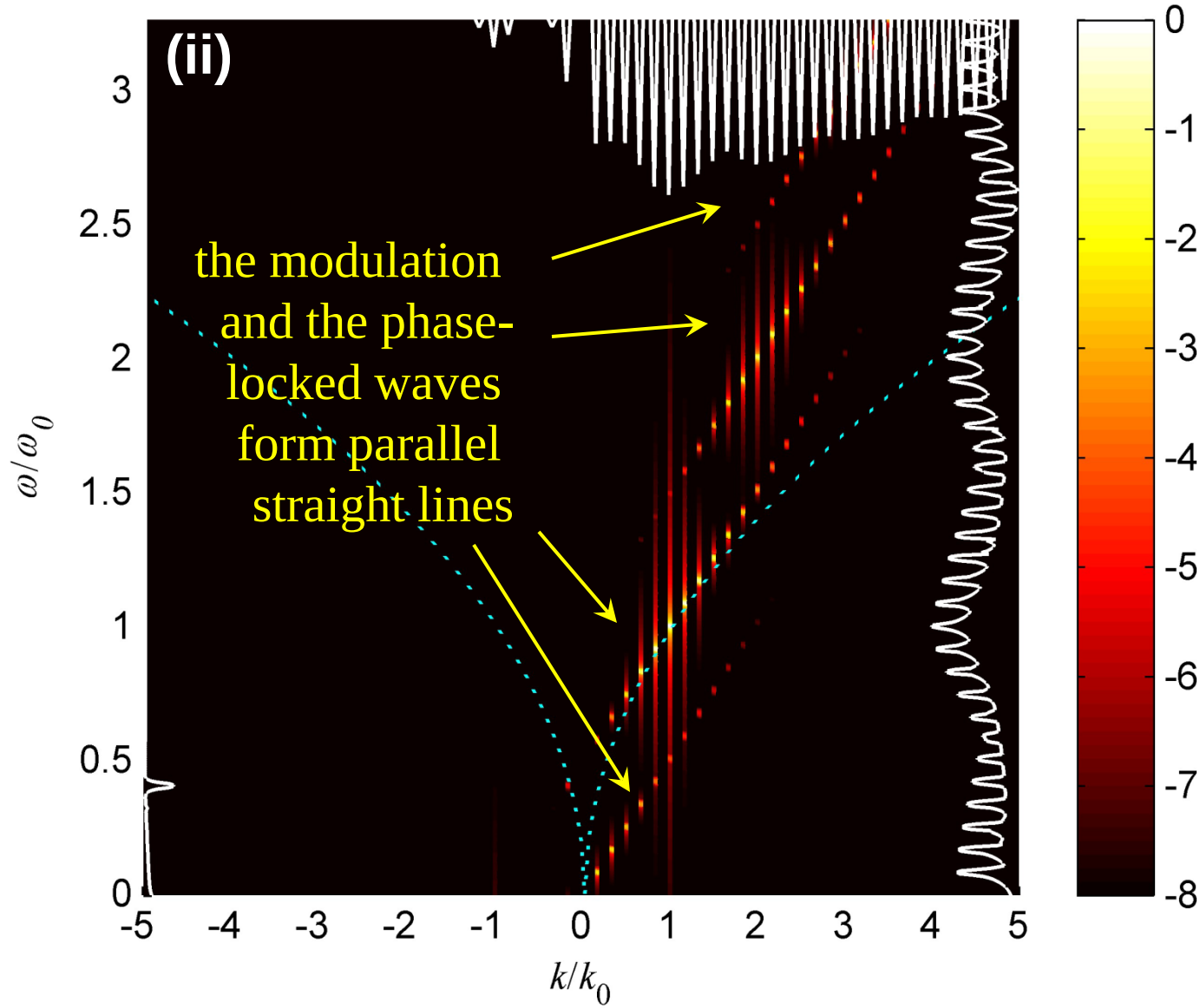
The modulational instability: the focusing event



$\varepsilon = 0.07$, $L/\lambda = 6$

Double Fourier transform plane

The modulational instability: the focusing event

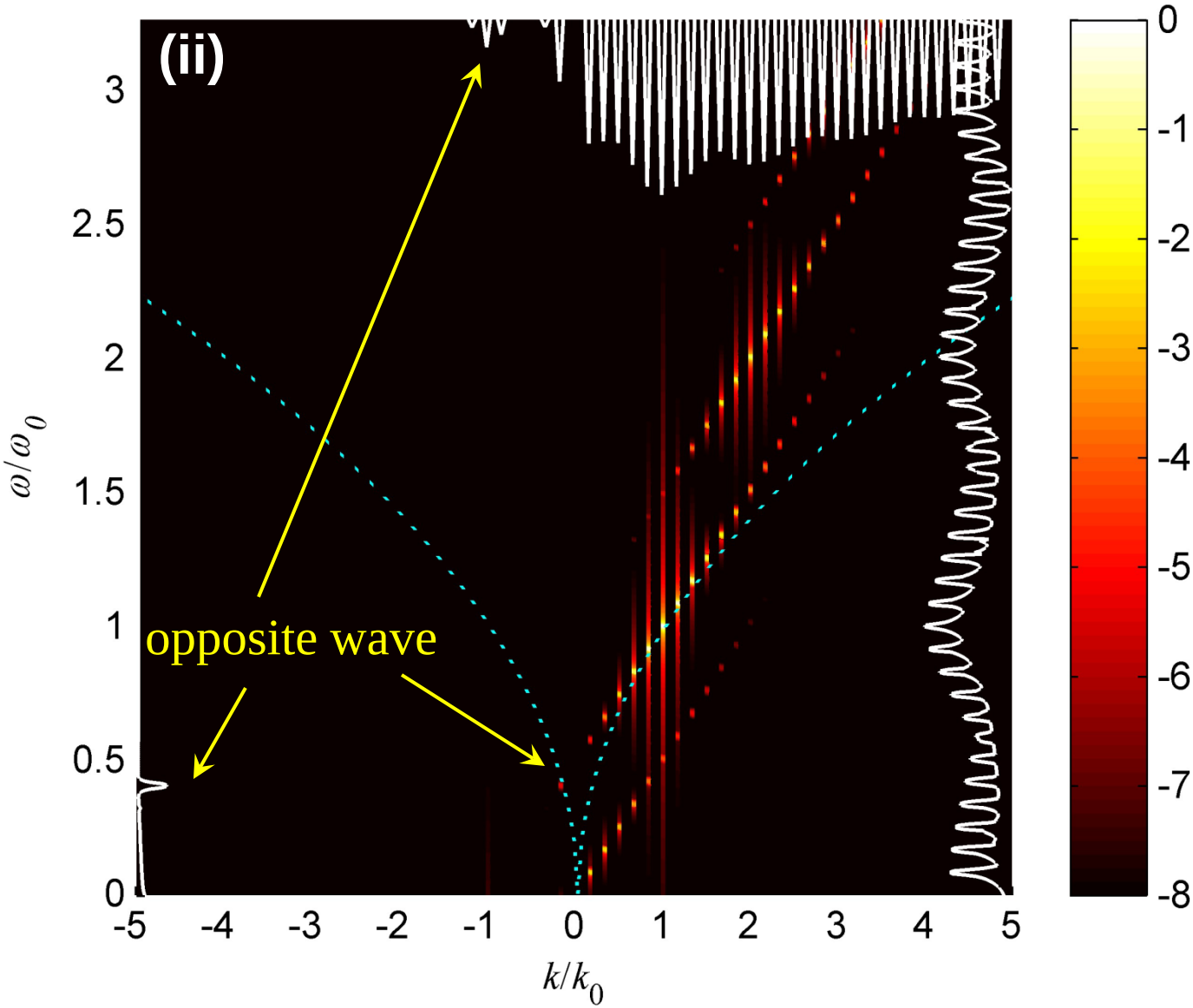


$\varepsilon = 0.07$, $L/\lambda = 6$

Appearance of a coherent strongly modulated group of strongly nonlinear waves

Double Fourier transform plane

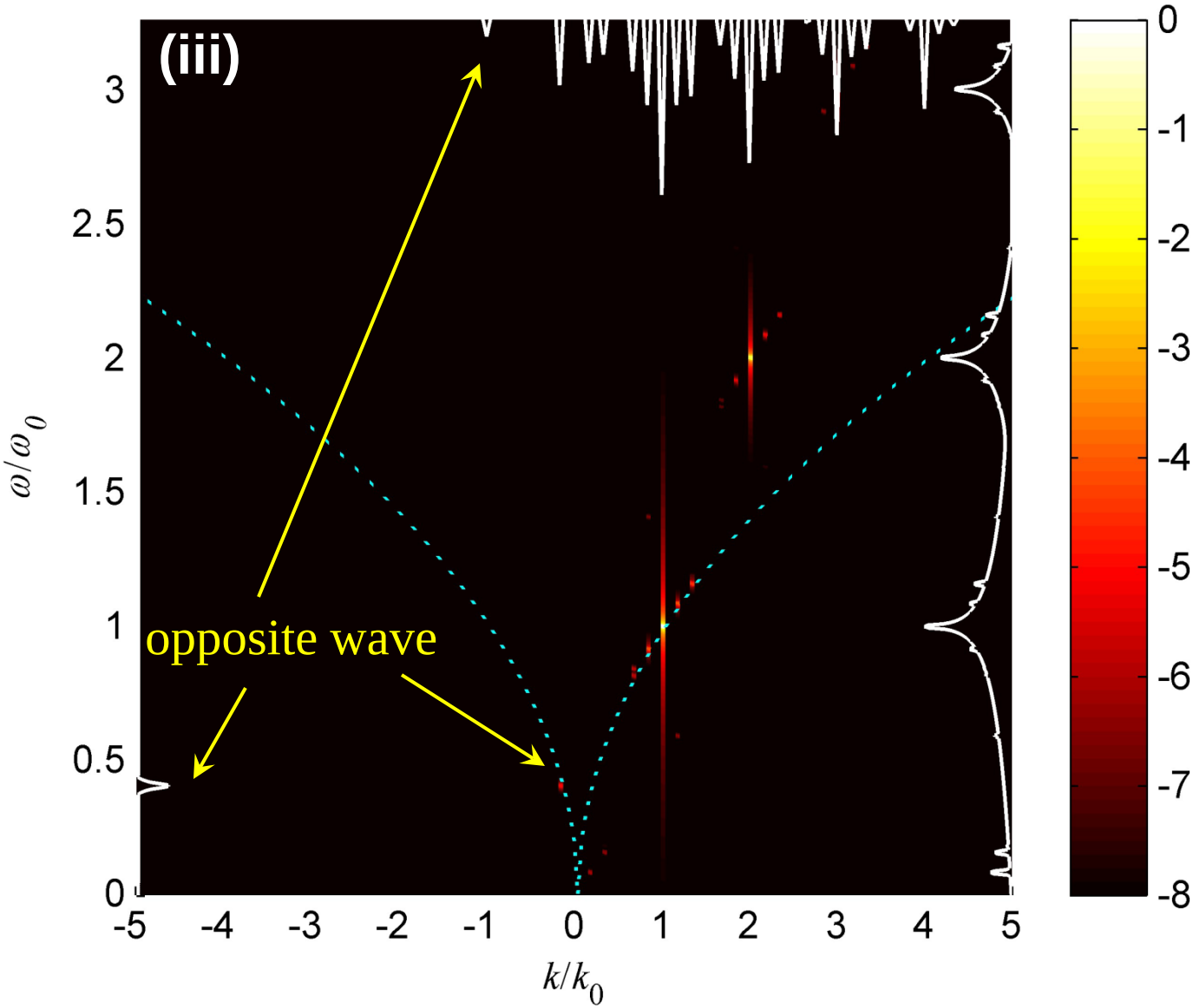
The modulational instability: the focusing event



$\varepsilon = 0.07$, $L/\lambda = 6$

Double Fourier transform plane

The modulational instability: demodulation stage



$\varepsilon = 0.07$, $L/\lambda = 6$

The modulation has relaxed, but the new wave component remains – the free (true) wave

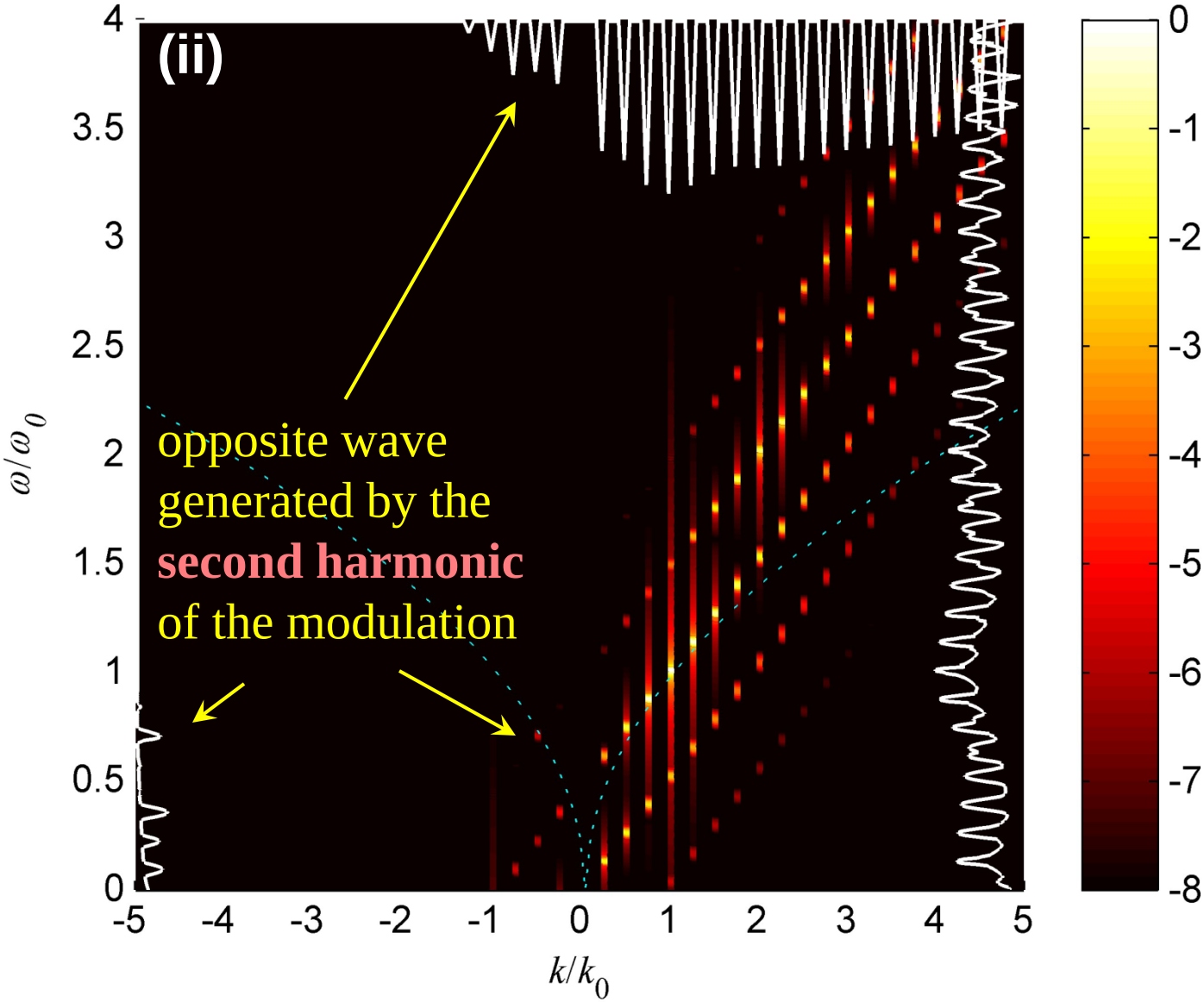
Nonlinear modulated waves

The evolution

$\varepsilon = 0.07$, $L/\lambda = 6$

Double Fourier transform plane

The modulational instability: the focusing event

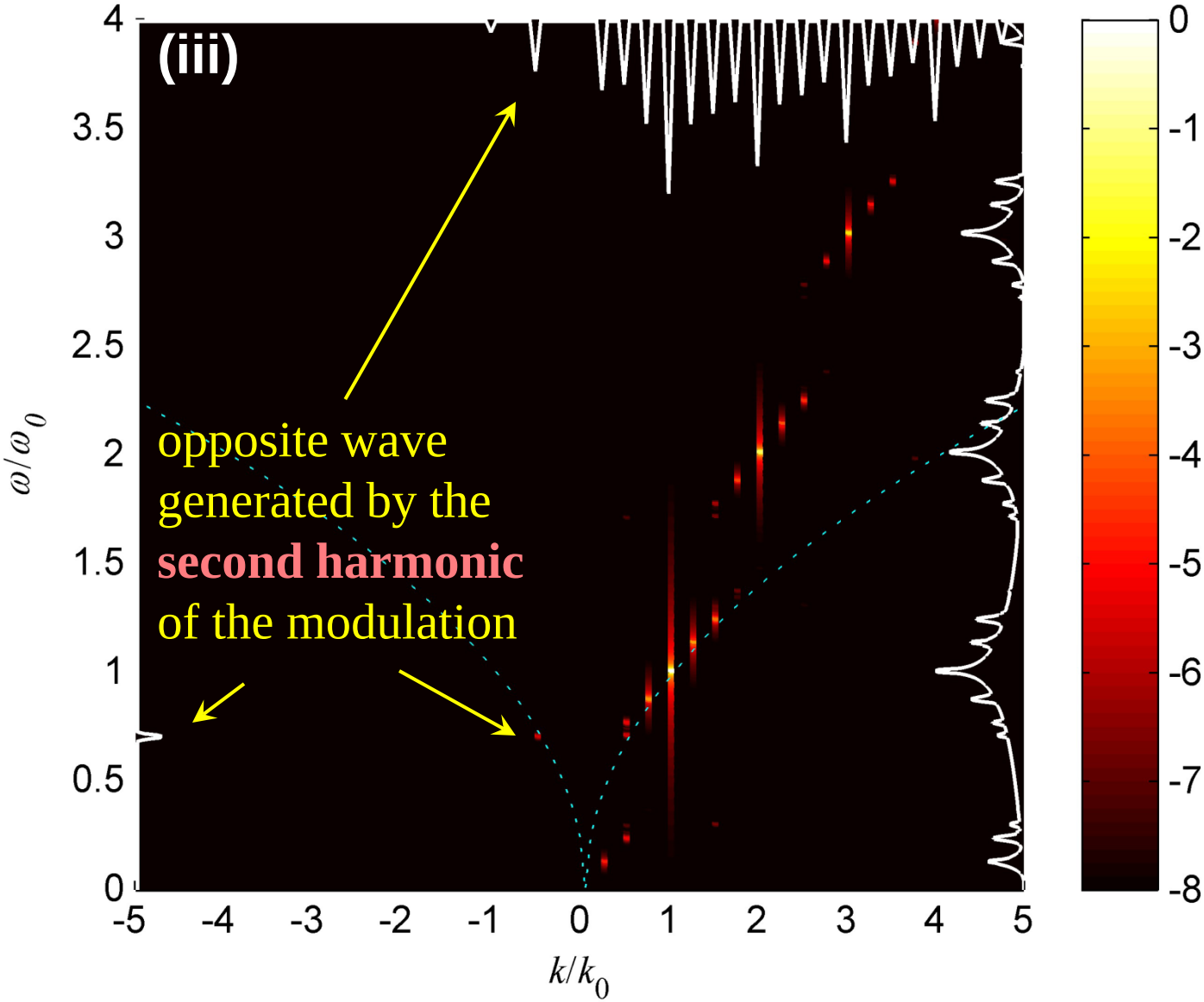


$\varepsilon = 0.11$, $L/\lambda = 4$ (different initial condition)

The bound wave component resulted in an occurrence of a new ('true') wave

Double Fourier transform plane

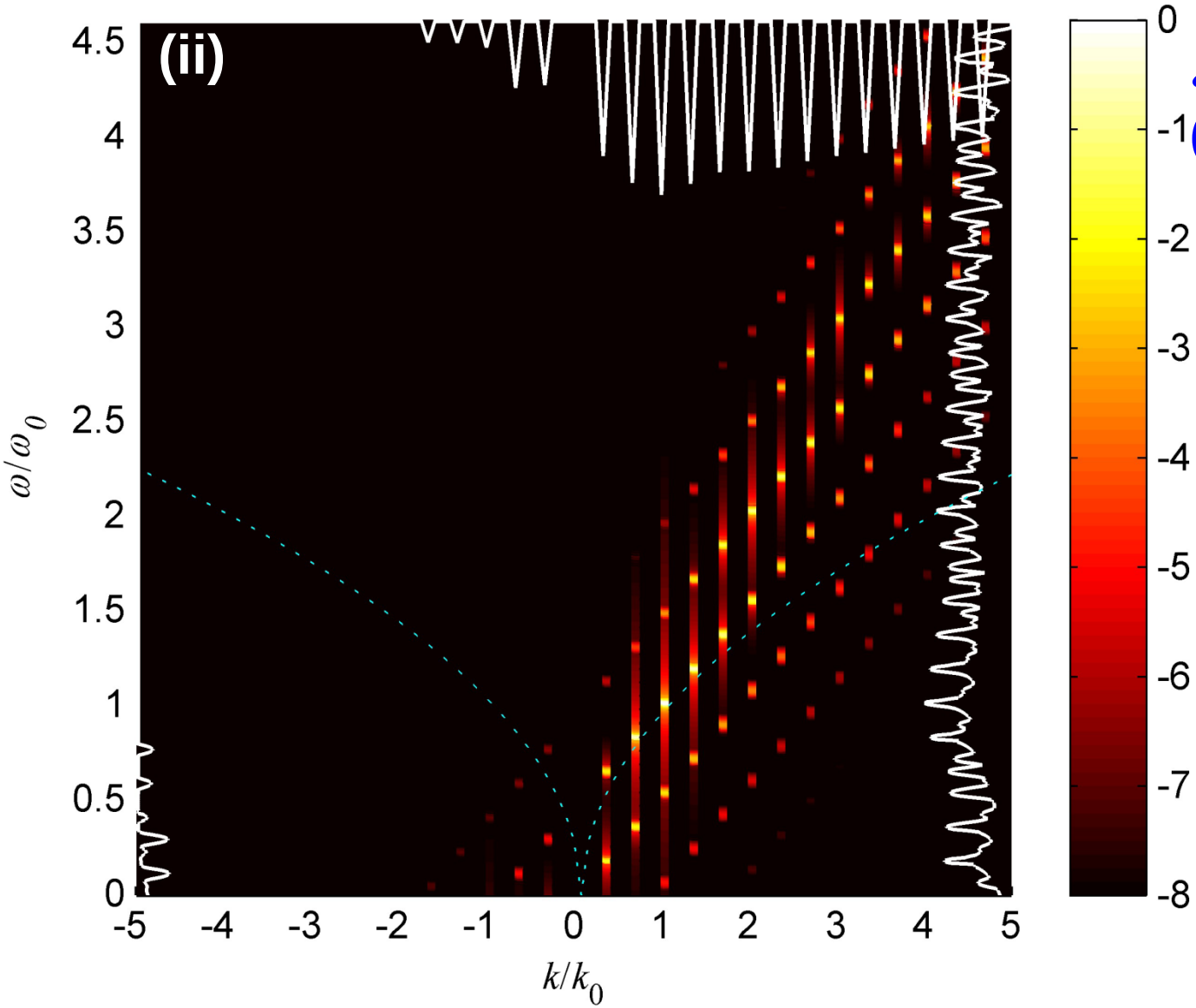
The modulational instability: demodulation stage



$\varepsilon = 0.11$, $L/\lambda = 4$

Double Fourier transform plane

The modulational instability: the focusing event

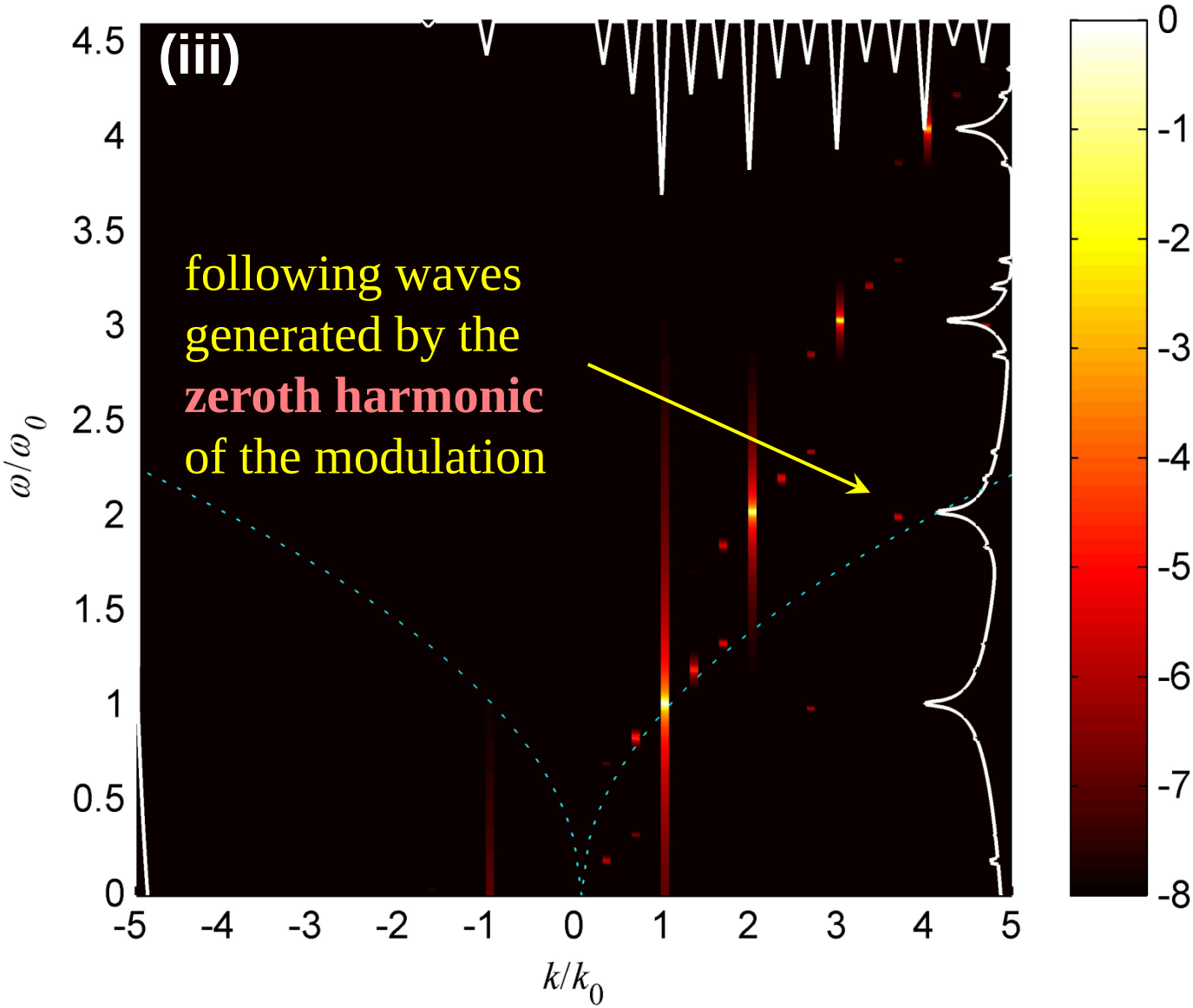


$\varepsilon = 0.146$, $L/\lambda = 3$

(different Initial condition)

Double Fourier transform plane

The modulational instability: demodulation stage



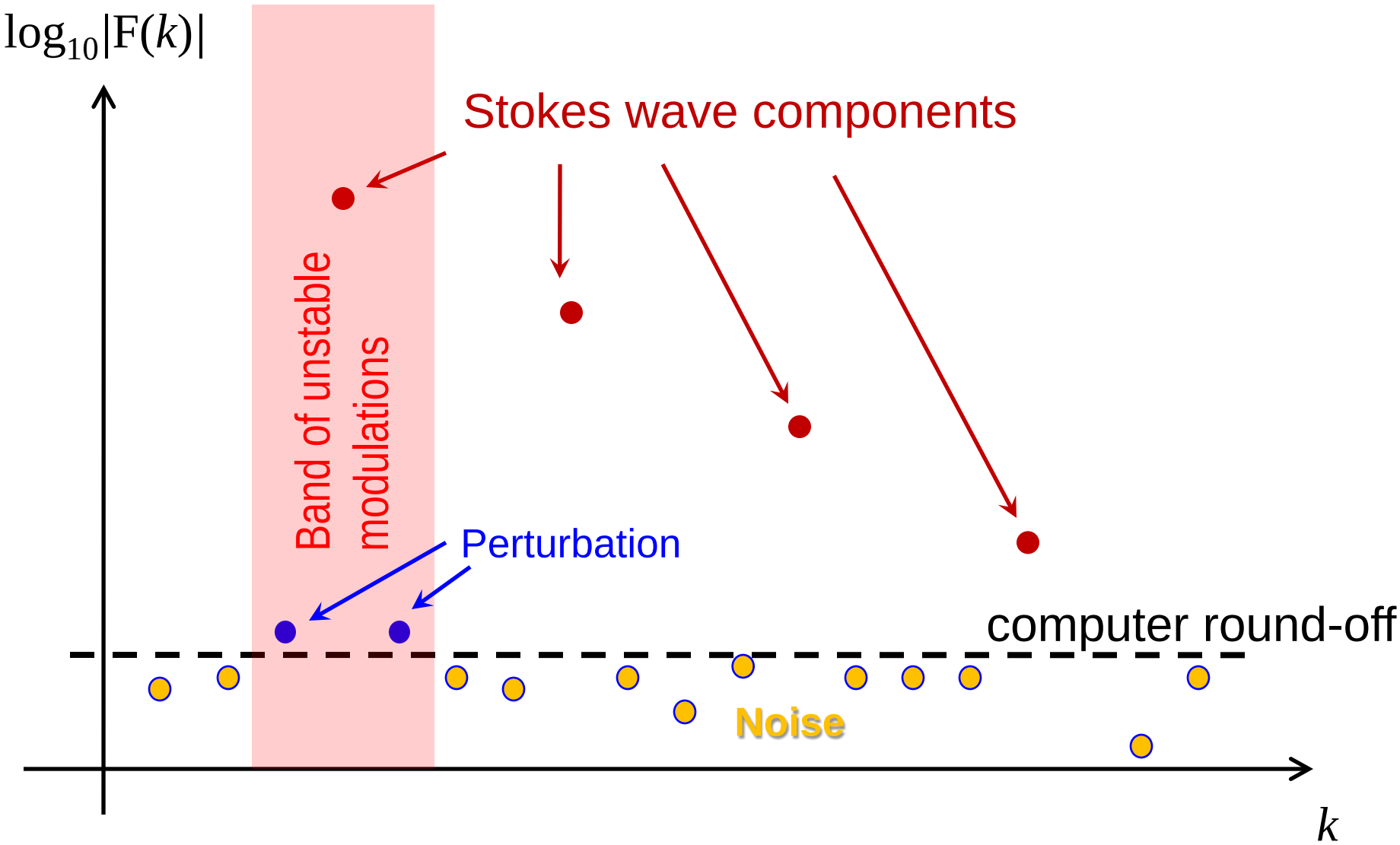
$\varepsilon = 0.146$, $L/\lambda = 3$

The bound wave component resulted in an occurrence of a new ('true') wave

This wave cannot be distinguished with the help of a 1D - spatial or temporal Fourier transform!

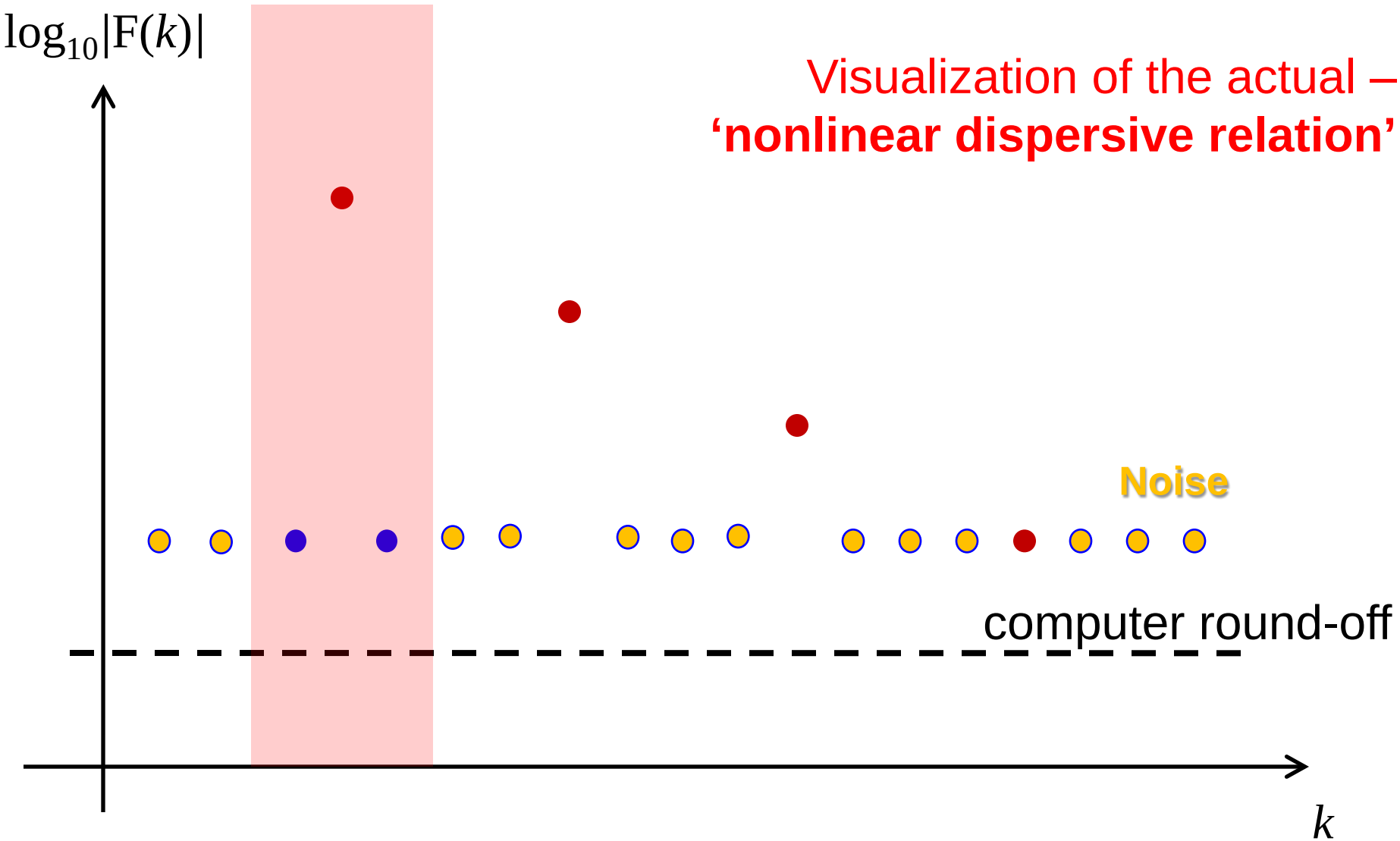
Nonlinear modulated waves

The 'nonlinear dispersive' relation: the initial condition



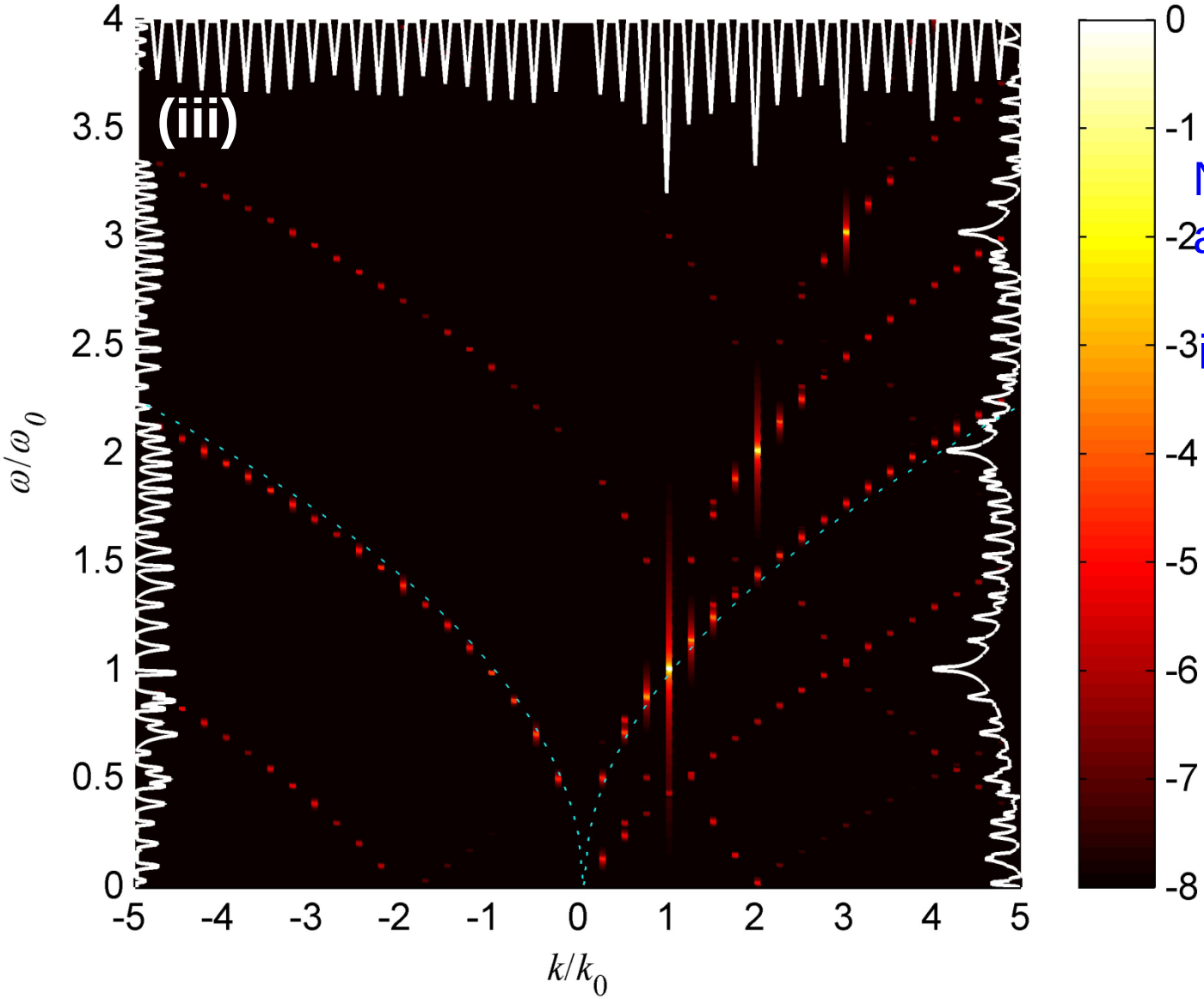
Nonlinear modulated waves

The 'nonlinear dispersive' relation: the initial condition



Nonlinear modulated waves

Visualization of the dispersive relation



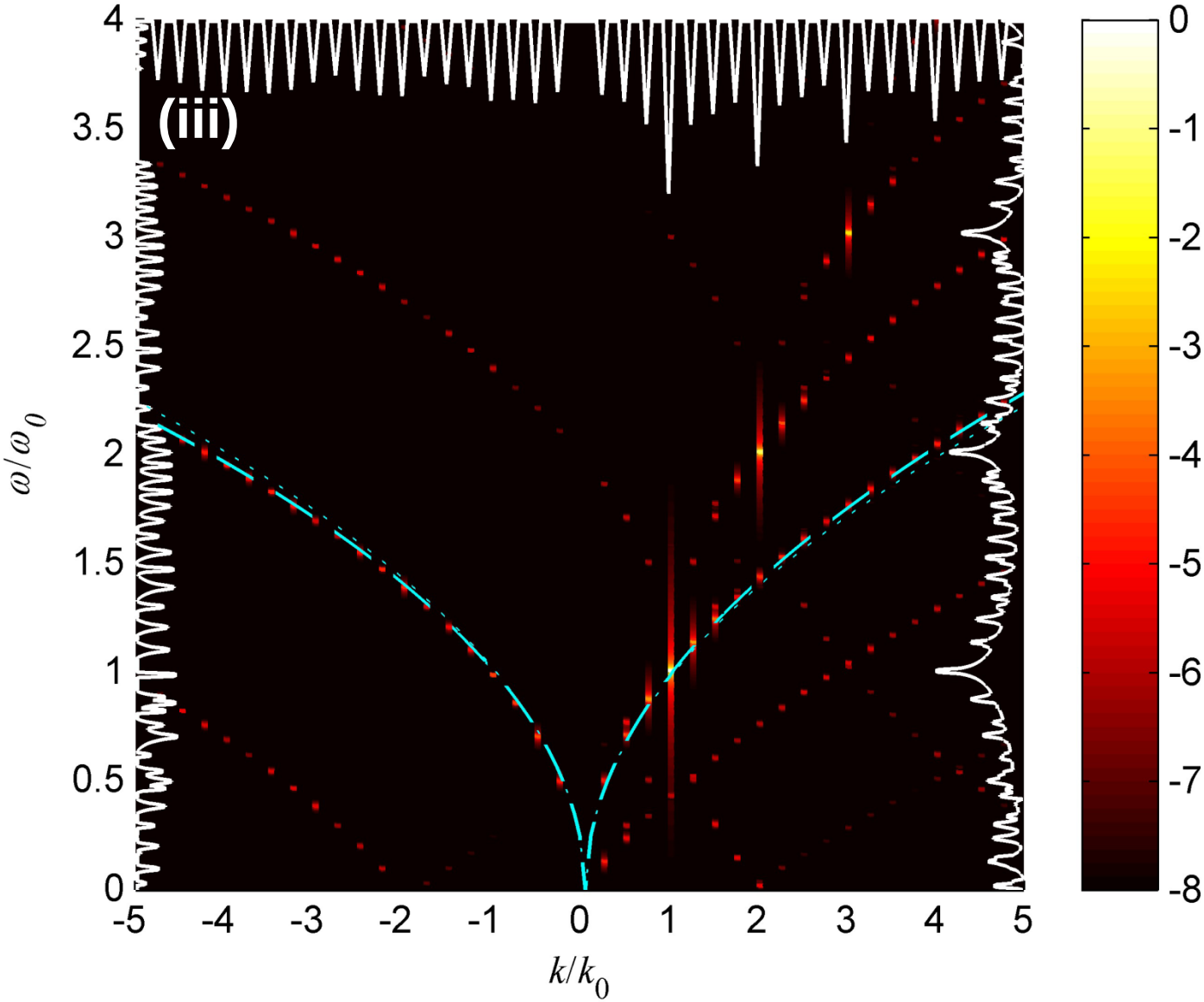
$\varepsilon = 0.11$, $L/\lambda = 4$

Noisy perturbations at all wavenumbers were introduced initially to make the actual dispersive relation visible

The excursion from the linear dispersive curve is obvious, especially in the short-wave bound

Nonlinear modulated waves

Visualization of the dispersive relation



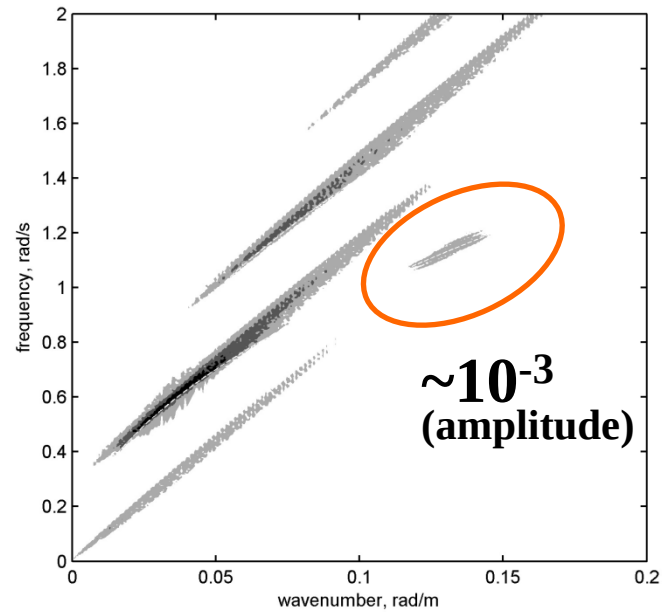
$\varepsilon = 0.11$, $L/\lambda = 4$

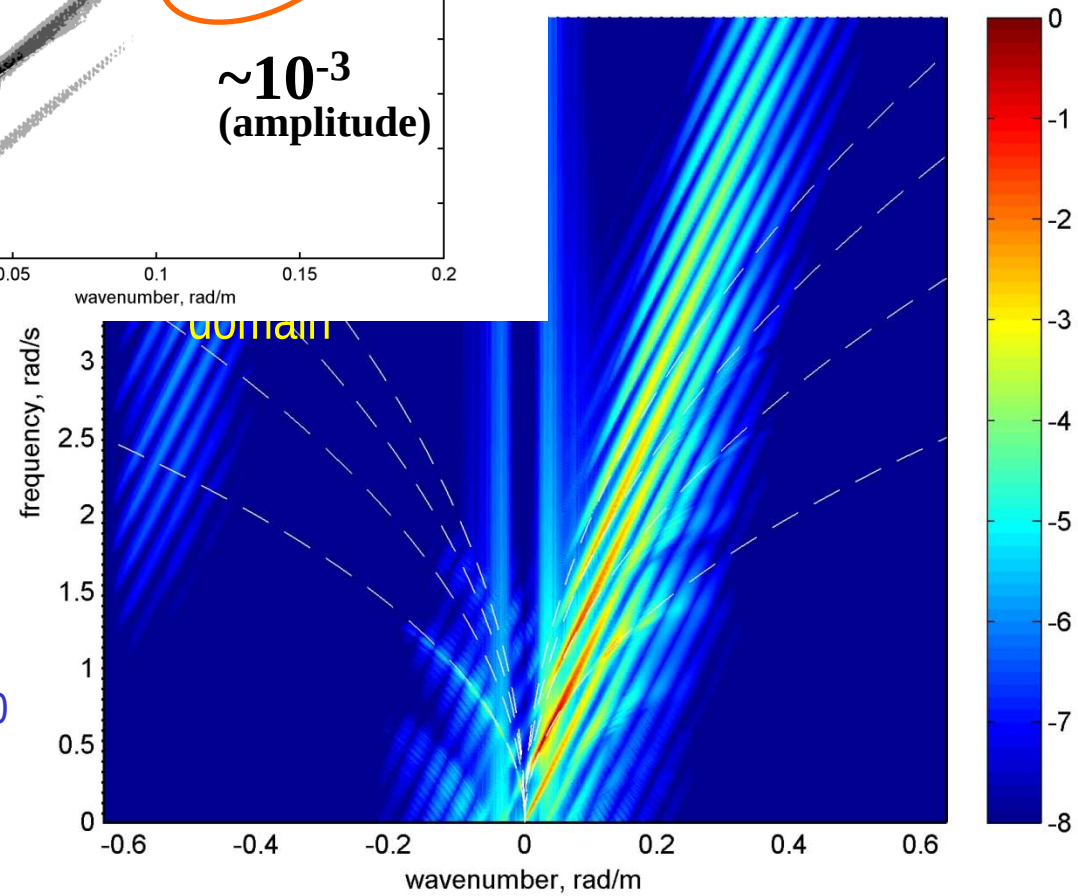
The actual dispersive relation may be described with the help of a modified dependence obtained within the weakly nonlinear theory for interacting waves (the long-dashed curve)

Spectra of nonlinear wave systems

Narrow Gaussian spectrum (more intense), HOSM (M = 6)

$\boldsymbol{H_s}$ **= 7 m**, T_p = 10.5 s,
$\Delta\omega/\omega = 0.1$, $\varepsilon_s = 0.13$





Excitation of short waves
by the nonlinear induced flow

Induced flow: $\omega = C_{gr}k$,
linear dispersion: $\omega^2 = gk$,
then resonance occurs at $k = 4k_0$

Cannot be found in spatial
or frequency spectra!

Conclusions - Technical

Неаккуратное возбуждение простой нелинейной волны с помощью простой линейной приводит к очень «богатому» спектру Фурье и нежелательным физическим эффектам (напр., стоячие волны). Рецепт избавления от проблемы известен

Требование узкого спектра для описания модуляций волн должно выполняться в пространстве волновых чисел и частот, а не в каких-то его проекциях

Привычные операции с модулированными волнами могут давать «странные» физически значимые эффекты

Фурье анализ в пространстве-времени – мощный (но не всесильный) инструмент для исследования нелинейных процессов («скрытые» резонансы)

Conclusions - Physical

Описанные эффекты имеют универсальный характер

Спектр Фурье солитона огибающей совсем не следует дисперсионной кривой. Сильно модулированные когерентные группы даже из слабонелинейных волн радикально меняют картину волновых резонансов (квазирезонансов)

Сильно нелинейные волны с одной длиной изменяют дисперсионные свойства волн с другой длиной

Связанные нелинейные компоненты – не обязательно «рабы» своих родителей – свободных волн (степеней свободы), они могут являться причиной динамических эффектов: приводить к генерации новых свободных волн и «обогащать» спектр

Утверждение, что однонаправленные волны на глубокой воде не рождают встречных волн – не совсем верно

Бризеры в исходных уравнениях гидродинамики (~ дышащие сепаратрисные решения для модуляций) не существуют с математической точки зрения

A.V. Slunyaev, Group-wave resonances in nonlinear dispersive media. *Phys. Rev. E* 97, 010202(R) (2018)

A. Slunyaev, A. Dosaev, On the incomplete recurrence of modulationally unstable deep-water surface gravity waves. arXiv: 1710.01477

Thank you very much for your attention!