Synchronization in Ensembles of Oscillators: Theory of Collective Dynamics

A. Pikovsky

Institut for Physics and Astronomy, University of Potsdam, Germany
Institute for Supercomputing, Nizhny Novgorod University, Russia

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- Populations of dynamical systems
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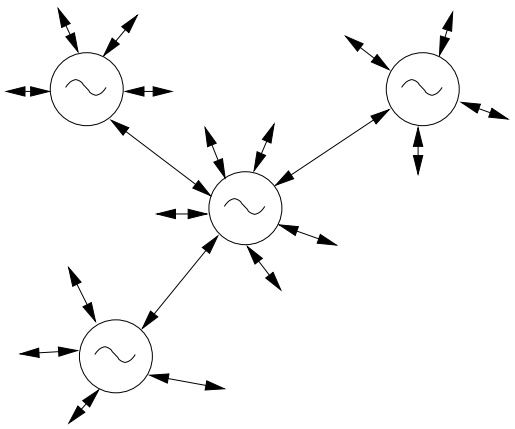
Ensembles of globally (all-to-all) couples oscillators

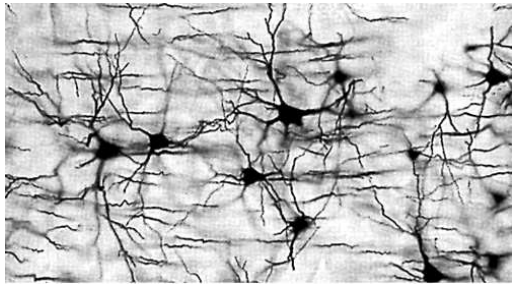
- Physics: arrays of spin-torque oscillators, Josephson junctions, multimode lasers,...
- Biology and neuroscience: cardiac pacemaker cells, population of fireflies, neuronal ensembles...
- Social behavior: applause in a large audience, pedestrians on a bridge,...



Main effect: Synchronization

Mutual coupling adjusts phases of indvidual systems, which start to keep pace with each other

Synchronization can be treated as a nonequilibrium phase transition!



Rather general formulation

$$\frac{d}{dt}\vec{x}_k = \vec{f}(\vec{x}_k, \vec{X}, \vec{Y}) \qquad \text{individual oscillators or other dynamical objects (}$$

$$\vec{X} = \frac{1}{N}\sum_k \vec{g}(\vec{x}_k) \qquad \text{mean fields (generalizations possible)}$$

$$\frac{d}{dt}\vec{Y} = \vec{h}(\vec{X}, \vec{Y}) \qquad \text{macroscopic global variables}$$

Typical setup for a synchronization problem:
$\vec{x}_k(t)$ – periodic or chaotic oscillators
$\vec{X}(t), \vec{Y}(t)$ periodic or chaotic $\Rightarrow$ collective synchronous rhythm
$\vec{X}(t), \vec{Y}(t)$ stationary $\Rightarrow$ desynchronization

Thermodynamic limit

In the limit $N \to \infty$ one can describe the population via the distribution density that obeys the Liouville equation

$$\frac{\partial}{\partial t}\rho(\vec{x},t) + \frac{\partial}{\partial \vec{x}}\left[\rho(\vec{x},t)\vec{f}(\vec{x},\vec{X},\vec{Y})\right] = 0$$

and the mean fields are

$$\vec{X}(t) = \int d\vec{x}\, \rho(\vec{x},t)\vec{g}(\vec{x})$$

The resulting system of nonlinear integro-differential equations is hard to study

Description in terms of macroscopic variables

The goal is to describe the ensemble in terms of macroscopic variables $\vec{W}$, which characterize the distribution of $\vec{x}_k$,

$$\dot{\vec{W}} = \vec{q}(\vec{W}, \vec{Y}) \qquad \text{generalized mean fields}$$
$$\dot{\vec{Y}} = \vec{h}(\vec{X}(\vec{W}), \vec{Y}) \qquad \text{global variables}$$

as a possibly low-dimensional dynamical system
Below: how this program works for phase oscillators by virtue of the Watanabe-Strogatz and the Ott-Antonsen approaches

Phase reduction for periodic oscillators

On the limit cycle the phase is well-defined $\frac{d\varphi}{dt} = \omega_0$
One can extend the definition of the phase to the whole basin of attraction of the limit cycle:

$$\frac{d\mathbf{A}}{dt} = \mathbf{F}(\mathbf{A}, \varphi) \qquad \frac{d\varphi}{dt} = \omega_0$$

Here $\mathbf{A}$ is "amplitude" which is stable, while the phase φ is marginally stable
As we know the phase on the limit cycle and close to it $\varphi(\mathbf{x})$, we get a closed equation for the phase, by substituting in the 1st order $\mathbf{x} \approx \mathbf{x}_0$:

$$\frac{d\varphi}{dt} = \frac{\partial\varphi}{\partial\mathbf{x}}\frac{d\mathbf{x}}{dt} \approx \frac{\partial\varphi}{\partial\mathbf{x}}\left[\mathbf{F}(\mathbf{x}_0) + \varepsilon\mathbf{P}(\mathbf{x}_0, t)\right] =$$
$$= \omega_0 + \varepsilon\frac{\partial\varphi}{\partial\mathbf{x}}(\mathbf{x}_0)\mathbf{P}(\mathbf{x}_0, t) = \omega_0 + \varepsilon Q(\varphi, t)$$

Coupling and averaging of the phase dynamics

If the forcing is from another oscillator with phase ψ, then we have $\mathbf{P}(\mathbf{x}, \psi)$ and the coupling equation

$$\frac{d\varphi}{dt} = \omega_0 + \varepsilon Q(\varphi, \psi)$$

Additional small parameter $1/\omega_0$: fast, compared to the time scale $1/\varepsilon$, oscillations
Averaging close to the main resonance $\frac{d}{dt}\varphi \approx \frac{d}{dt}\psi$
Because $Q(\varphi, \psi)$ is 2π-periodic in both arguments, use double Fourier representation $Q(\varphi, \psi) = \sum_{m,l} Q_{m,l} \exp[im\varphi - il\psi]$ and keep only terms with $l = m$:

$$\frac{d\varphi}{dt} = \omega_0 + \varepsilon q(\varphi - \psi)$$

Typical coupling function: $q(\varphi - \psi) = \sin(\varphi - \psi - \beta)$

Kuramoto model: coupled phase oscillators

Phase oscillators ($\varphi_k \sim x_k$) with all-to-all pair-wise coupling

$$\dot{\varphi}_k = \omega_k + \varepsilon \frac{1}{N} \sum_{j=1}^{N} \sin(\varphi_j - \varphi_k + \beta)$$

$$= \varepsilon \left[\frac{1}{N} \sum_{j=1}^{N} \sin \varphi_j \right] \cos(\varphi_k - \beta) - \varepsilon \left[\frac{1}{N} \sum_{j=1}^{N} \cos \varphi_j \right] \sin(\varphi_k - \beta)$$

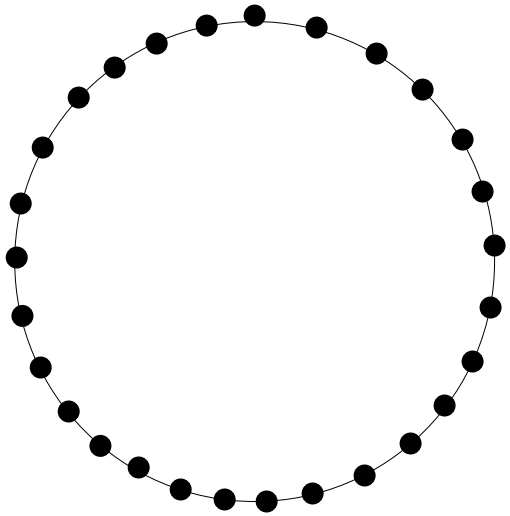
$$= \omega_k + \varepsilon R(t) \sin(\Theta(t) - \varphi_k - \alpha) = \omega_k + \varepsilon \mathrm{Im}(Z e^{-i\varphi_k + i\beta})$$

System can be written as a mean-field coupling with the mean field (**complex order parameter** $Z \sim X$)

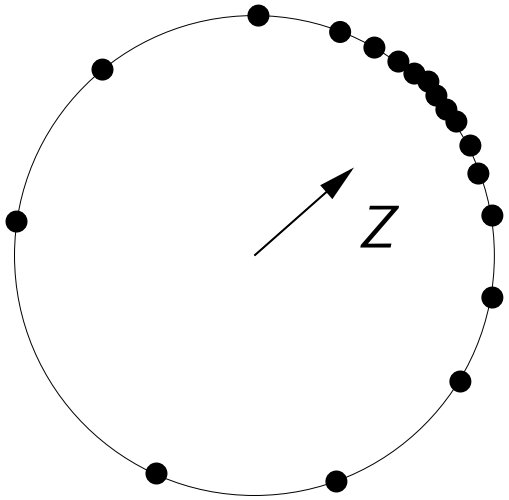
$$Z = R e^{i\Theta} = \frac{1}{N} \sum_k e^{i\varphi_k}$$

Synchronisation transition

$\varepsilon_c \sim$ width of distribution of frequecies $g(\omega) \sim$ "temperature"



small ε: no synchronization, phases are distributed uniformly, mean field vanishes $Z = 0$



large ε: synchronization, distribution of phases is non-uniform, finite mean field $Z \neq 0$

Ott-Antonsen ansatz

[E. Ott and T. M. Antonsen, CHAOS 18 (037113) 2008]

Consider the same system

$$\frac{d\varphi_k}{dt} = \omega(t) + \mathrm{Im}\left(H(t)e^{-i\varphi_k}\right) \qquad k = 1, \ldots, N$$

in the thermodynamic limit $N \to \infty$ and write equation for the probability density $\rho(\varphi, t)$:

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial \varphi}\left[\rho\left(\omega + \frac{1}{2\mathrm{i}}(He^{-\mathrm{i}\varphi} - H^* e^{\mathrm{i}\varphi})\right)\right] = 0$$

Expanding density in Fourier modes $\rho = (2\pi)^{-1}\sum W_k(t)e^{-\mathrm{i}k\varphi}$ yields an infinite system

$$\frac{dW_k}{dt} = \mathrm{i}k\omega W_k + \frac{k}{2}(HW_{k-1} - H^* W_{k+1})$$

$$\frac{dW_1}{dt} = \mathrm{i}\omega W_1 + \frac{1}{2}(H - H^* W_2)$$

$$\frac{dW_k}{dt} = \mathrm{i}k\omega W_k + \frac{k}{2}(HW_{k-1} - H^* W_{k+1})$$

With an ansatz $W_k = (W_1)^k$ we get for $k \geq 2$

$$\frac{dW_k}{dt} = kW_1^{k-1}\left[\mathrm{i}\omega W_1 + \frac{1}{2}(H - H^* W_1^2)\right]$$

ie all the infite system is reduced to one equation.

OA equation for the Kuramoto model

Because $W_1 = \langle e^{\mathrm{i}\varphi} \rangle = Z$ we get the Ott-Antonsen equation

$$\frac{dZ}{dt} = \mathrm{i}\omega Z + \frac{1}{2}(H - H^* Z^2)$$

The forcing in the Kuramoto-Sakaguchi model is due to the mean field $H = Ze^{i\beta}$

One obtains a closed equation fro the dynamics of the mean field:

$$\frac{dZ}{dt} = i\omega Z + \frac{\varepsilon}{2} e^{i\beta} Z - \frac{\varepsilon}{2} e^{-i\beta} |Z|^2 Z$$

Closed equation for the real order parameter $R = |Z|$:

$$\frac{dR}{dt} = \frac{\varepsilon}{2} R(1 - R^2) \cos\beta$$

Simple dynamics in the Kuramoto-Sakaguchi model

$$\frac{dR}{dt} = \frac{\varepsilon}{2} R(1 - R^2) \cos\beta$$

Attraction: $-\frac{\pi}{2} < \beta < \frac{\pi}{2}$ $\Longrightarrow$
Synchronization, all phases identical $\varphi_1 = \ldots = \varphi_N$, order parameter large $R = 1$
Repulsion: $-\pi < \beta < -\frac{\pi}{2}$ and $\frac{\pi}{2} < \beta < \pi$ $\Longrightarrow$
Asynchrony, phases distributed uniformely, order parameter vanishes $R = 0$

Application to nonidentical oscillators

Assuming a distribution of natural frequencies $g(\omega)$, one introduces $Z(\omega) = \rho(\omega)e^{i\Phi(\omega)}$ and obtains the Ott-Antonsen integral equations

$$\frac{\partial Z(\omega, t)}{\partial t} = i\omega Z + \frac{1}{2}Y - \frac{Z^2}{2}Y^*$$

$$Y = e^{i\beta}\langle e^{\mathrm{i}\varphi}\rangle = e^{i\beta}\int d\omega\, g(\omega)Z(\omega)$$

OA equations for Lorentzian distribution of frequencies

If

$$g(\omega) = \frac{\Delta}{\pi((\omega - \omega_0)^2 + \Delta^2)}$$

and Z has no poles in the upper half-plane, then the integral $Y = \int d\omega\, g(\omega) Z(\omega)$ can be calculated via residues as $Y = Z(\omega_0 + i\Delta)$

This yields an ordinary differential equation for the order parameter Y

$$\frac{dY}{dt} = (i\omega_0 - \Delta)Y + \frac{1}{2}\varepsilon(e^{i\beta} - e^{-i\beta}|Y|^2)Y$$

Hopf normal form / Landau-Stuart equation/ Poincaré oscillator

$$\frac{dY}{dt} = (a + ib - (c + id)|Y|^2)Y$$

Summary of OA ansatz

- An invariant parametrization of the distribution density - OA invariant manifold
- Stability has been claimed for non-identical oscillators
- Valid in thermodynamic limit only
- Restricted to pure sine-coupling

Watanabe-Strogatz (WS) ansatz

[S. Watanabe and S. H. Strogatz, PRL 70 (2391) 1993; Physica D 74 (197) 1994]

Ensemble of **identical** oscillators driven by the same complex field $H(t)$ and the real field $\omega(t)$

$$\frac{d\varphi_k}{dt} = \omega(t) + \mathrm{Im}\left(H(t)e^{-i\varphi_k}\right) \qquad k = 1, \ldots, N$$

This equation also describes the dynamics of the rear wheel of a bicycle if the front one is driven

Möbius transformation

Rewrite equation as

$$\frac{d}{dt}e^{\mathrm{i}\varphi_k} = \mathrm{i}\omega_k(t)e^{\mathrm{i}\varphi_k} + \frac{1}{2}H(t) - \frac{e^{\mathrm{i}2\varphi_k}}{2}H^*(t)$$

Möbius transformation from N variables φ_k to complex $z(t)$, $|z| \leq 1$, and N new angles $\psi_k(t)$, according to

$$e^{\mathrm{i}\varphi_k} = \frac{z + e^{\mathrm{i}\psi_k}}{1 + z^* e^{\mathrm{i}\psi_k}}$$

Since the system is over-determined, we require $N^{-1}\sum_{k=1}^{N} e^{\mathrm{i}\psi_k} = \langle e^{\mathrm{i}\psi_k} \rangle = 0$

WS equations

Direct substitution allows one (1 page calculation) to get WS equations

$$\dot{z} = \mathrm{i}\omega z + \frac{H}{2} - \frac{H^*}{2} z^2$$
$$\dot{\psi}_k = \omega + \mathrm{Im}(z^* H)$$

Remarkably: dynamics of ψ_k does not depend on k, thus introducing $\psi_k = \alpha(t) + \tilde{\psi}_k$ we get constants $\tilde{\psi}_k$ and 3 WS equations

$$\frac{dz}{dt} = i\omega z + \frac{1}{2}(H - z^2 H^*) \qquad \frac{d\alpha}{dt} = \omega + \mathrm{Im}(z^* H)$$

Three dynamical variables + $(N-3)$ integrals of motion

Interpretation of WS variables

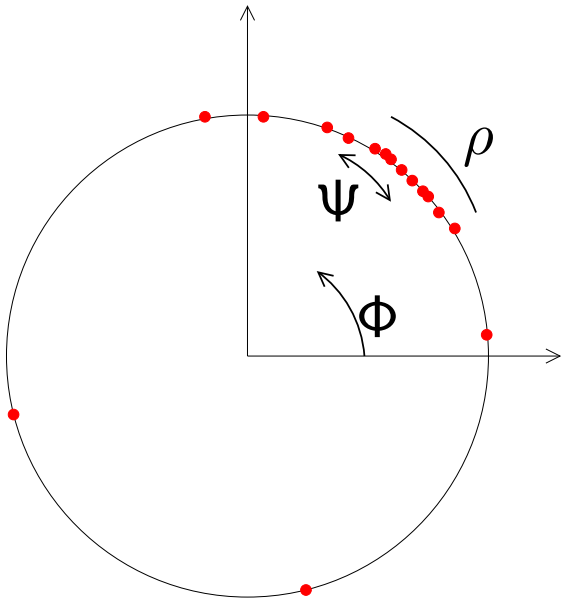
We write $z = \rho e^{i\Phi}$, then

$$e^{i\varphi_k} = e^{i\Phi(t)} \frac{\rho(t) + e^{i(\tilde{\psi}_k + \alpha(t) - \Phi(t))}}{\rho(t) e^{i(\tilde{\psi}_k + \alpha(t) - \Phi(t))} + 1}$$

ρ measures the width of the bunch:
$\rho = 0$ if the mean field $Z = \sum_k e^{i\varphi_k}$ vanishes
$\rho = 1$ if the oscillators are fully synchronized and $|Z| = 1$

Φ is the phase of the bunch

$\Psi = \alpha - \Phi$ measures positions of individual oscillators with respect to the bunch



Summary of WS transformations

- Works for a large class of initial conditions [does not work if the condition $\langle e^{\mathrm{i}\psi_k} \rangle = 0$ cannot be satisfied, eg if large clusters exist]
- Applies for any N, allows a thermodynamic limit where distribution of $\tilde{\psi}_k$ is constant in time, and only $z(t), \alpha(t)$ evolve
- Applies only if the r.h.s. of the phase dynamics contains 1st harmonics $\sin\varphi$, $\cos\varphi$
- Applies only if the oscillators are identical and identically driven

Complex order parameters in WS variables

Complex order parameter can be represented in WS variables as

$$Z = \sum_k e^{i\varphi_k} = \rho e^{i\Phi}\gamma(\rho, \Psi) \qquad \gamma = 1 + (1-\rho^{-2})\sum_{l=2}^{\infty} C_l(-\rho e^{-i\Psi})^l$$

where $C_l = N^{-1}\sum_k e^{il\psi_k}$ are Fourier harmonics of the distribution of constants ψ_k

Important simplifying case:

Uniform distribution of constants ψ_k

$$C_l = 0 \quad \Rightarrow \quad \gamma = 1 \quad \Rightarrow \quad Z = \rho e^{i\Phi} = z$$

In this case WS variables yield the order parameter directly and the WS equations are the OA equations

Relation WS $\leftrightarrow$ OA

- OA is the same as WS for $N \to \infty$ and for the uniform distribution of constants ψ_k
- A special familly of distributions satisfying $W_k = (W_1)^k$ is called OA manifold, it corresponds to all possible Möbius transformation of the uniform density of constants
- OA is formulated directly in terms of the Kuramoto order parameter
- For identical oscillators OA manifold is not attractive, but neutral

Two main types of synchronization

- Kuramoto-type synchronization:
  - Mean field is periodic
  - all or some oscillators are locked by the mean field
- Partial synchronization:
  - Mean field is periodic
  - oscillators are not locked by the mean field – quasiperiodic dynamics

Linear vs nonlinear coupling I

- Synchronization of a periodic autonomous oscillator is a nonlinear phenomenon
- it occurs already for infinitely small forcing
- because the unperturbed system is singular (zero Lyapunov exponent)

In the Kuramoto model "linearity" with respect to forcing is assumed

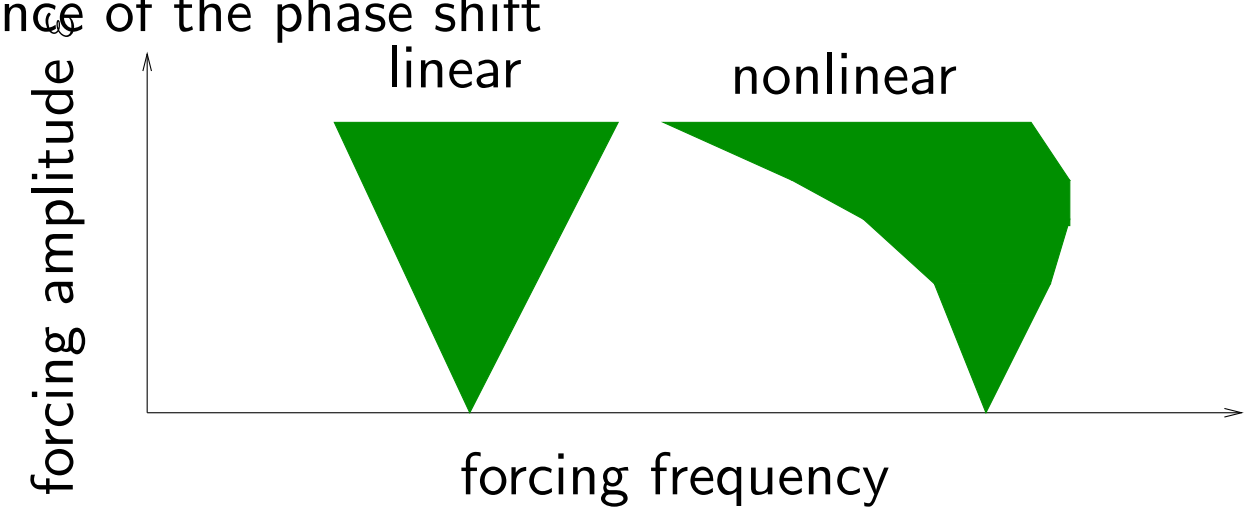
$$\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x}) + \varepsilon_1 \mathbf{f}_1(t) + \varepsilon_2 \mathbf{f}_2(t) + \cdots$$
$$\dot{\varphi} = \omega + \varepsilon_1 q_1(\varphi, t) + \varepsilon_2 q_2(\varphi, t) + \cdots$$

Linear vs nonlinear coupling II

Strong forcing leads to "nonlinear" dependence on the forcing amplitude

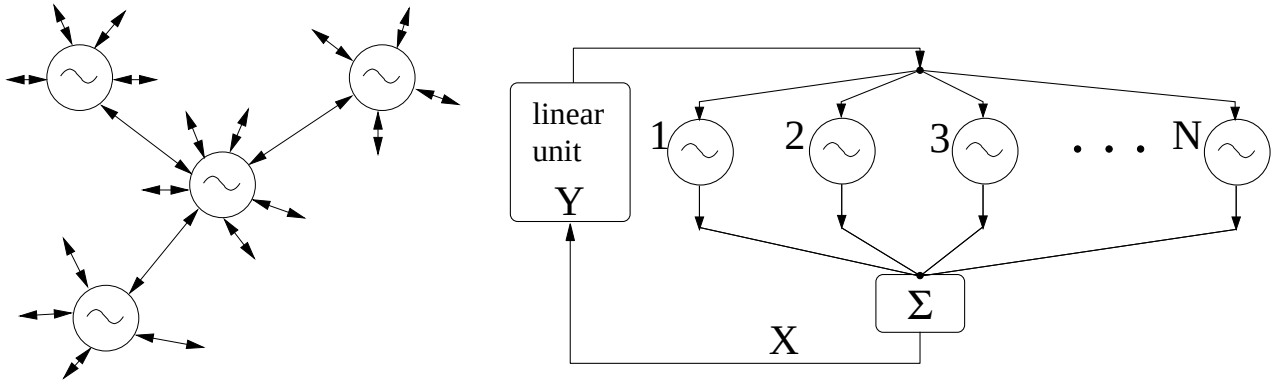
$$\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x}) + \varepsilon \mathbf{f}(t)$$
$$\dot{\varphi} = \omega + \varepsilon q^{(1)}(\varphi, t) + \varepsilon^2 q^{(2)}(\varphi, t) + \cdots$$

Nonlineraity of forcing manifests itself in the deformation/skeweness of the Arnold tongue and in the amplitude depnedence of the phase shift



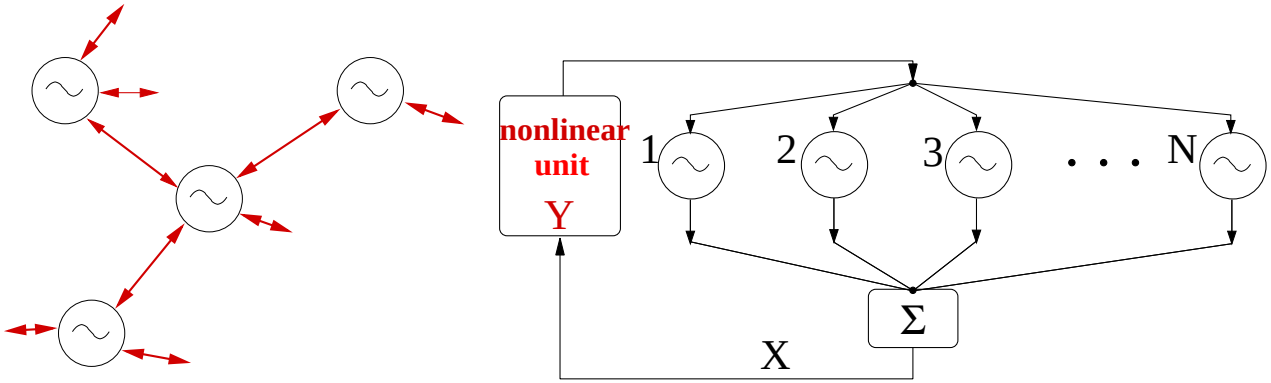
Linear vs nonlinear coupling III

Small each-to-each coupling $\Longleftrightarrow$ coupling via linear mean field



Strong each-to-each coupling $\Longleftrightarrow$ coupling via nonlinear mean field

[cf. Popovych, Hauptmann, Tass, Phys. Rev. Lett. 2005]



Nonlinear coupling: a minimal model

We take the standard Kuramoto-Sakaguchi model

$$\dot{\varphi}_k = \omega + \mathrm{Im}(He^{-i\varphi_k}) \qquad H \sim \varepsilon e^{-i\beta} Z \qquad Z = \frac{1}{N}\sum_j e^{i\varphi_j} = Re^{i\Theta}$$

and assume dependence of the acting force H on the "amplitude" of the mean field R:

$$\boxed{\dot{\varphi}_k = \omega + A(\varepsilon R)\varepsilon R \sin(\Theta - \varphi_k + \beta(\varepsilon R))}$$

E.g. attraction for small R vs repulsion for large R

WS/OA equations for the nonlinearly coupled ensemble

$$\frac{dR}{dt} = \frac{1}{2}R(1-R^2)\varepsilon A(\varepsilon R)\cos\beta(\varepsilon R)$$

$$\frac{d\Phi}{dt} = \omega + \frac{1}{2}(1+R^2)\varepsilon A(\varepsilon R)\sin\beta(\varepsilon R)$$

$$\frac{d\Psi}{dt} = \frac{1}{2}(1-R^2)\varepsilon A(\varepsilon R)\sin\beta(\varepsilon R)$$

Full vs partial synchrony

All regimes follow from the equation for the order parameter

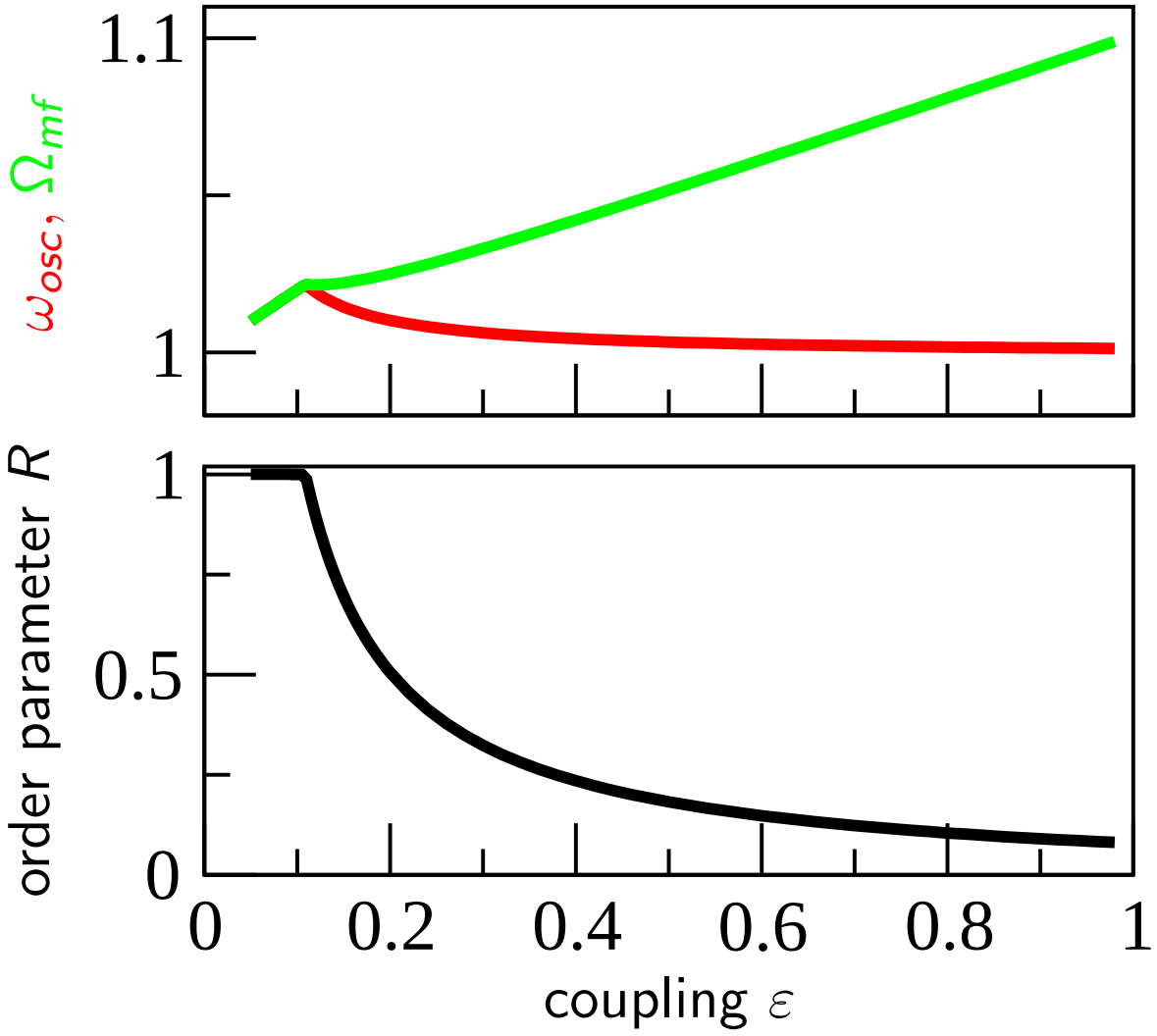
$$\frac{dR}{dt} = \frac{1}{2}R(1-R^2)\varepsilon A(\varepsilon R)\cos\beta(\varepsilon R)$$

Fully synchronous state: $R = 1,\ \dot{\Phi} = \omega + \varepsilon A(\varepsilon)\sin\beta(\varepsilon)$
Asynchronous state: $R = 0$
Partially synchronous bunch state
$0 < R < 1$ from the condition $A(\varepsilon R) = 0$:
No rotations, frequency of the mean field = frequency of the oscillations
Partially synchronized quasiperiodic state
$0 < R < 1$ from the condition $\cos\beta(\varepsilon R) = 0$:
Frequency of the mean field $\Omega = \dot{\varphi} = \omega \pm A(\varepsilon R)(1+R^2)/2$
Frequency of oscillators $\omega_{osc} = \omega \pm A(\varepsilon R)R^2$

Self-organized quasiperiodicity

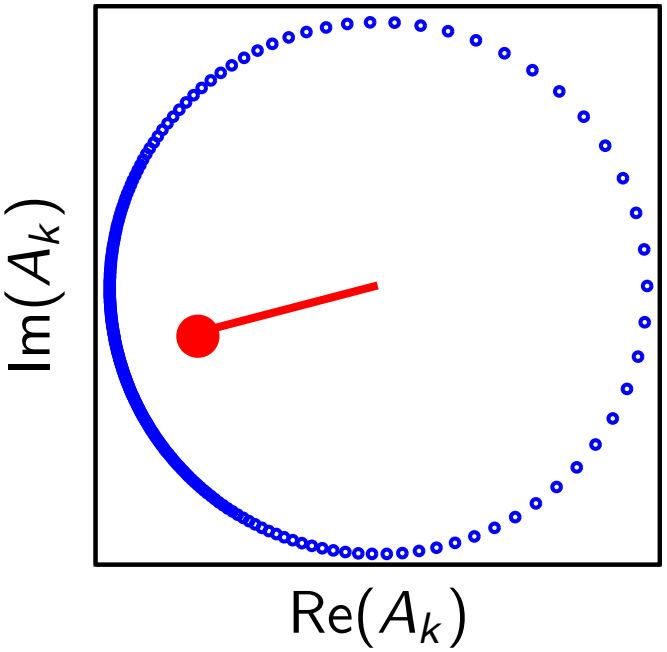
- frequencies Ω and ω_{osc} depend on ε in a smooth way
  $\Longrightarrow$ generally we observe a quasiperiodicity
- attraction for small mean field vs repulsion for large mean field
  $\Longrightarrow$ ensemble is always at the stabilty border
  $\beta(\varepsilon R) = \pm\pi/2$, i.e. in a
  self-organized critical state
- critical coupling for the transition from full to partial synchrony:
  $\beta(\varepsilon_q) = \pm\pi/2$
- transition at "zero temperature" like quantum phase transition

Simulation: loss of synchrony with increase of coupling



Simulation: snapshot of the ensemble

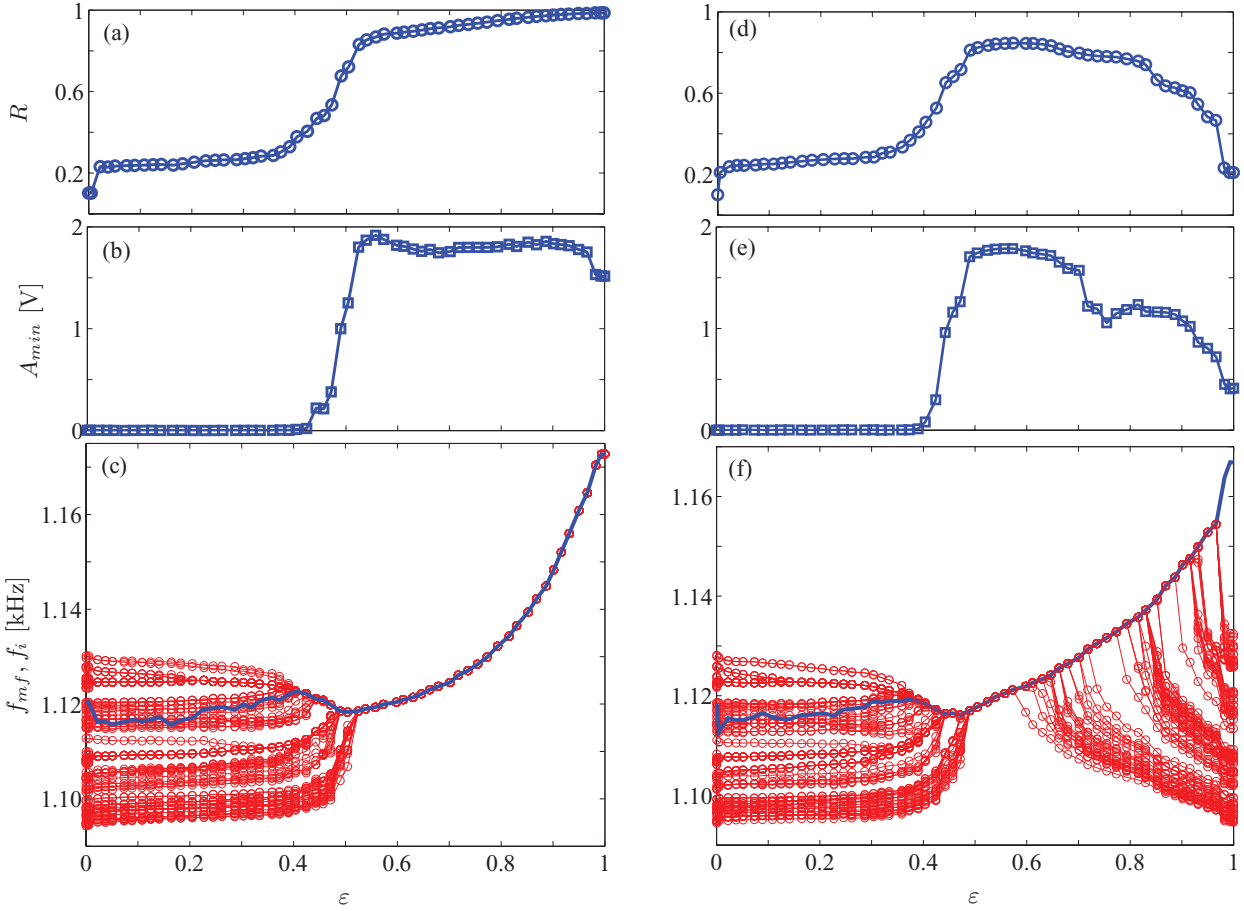
- non-uniform distribution of oscillator phases, here for $\varepsilon - \varepsilon_q = 0.2$
- different velocities of oscillators and of the mean field



Experiment

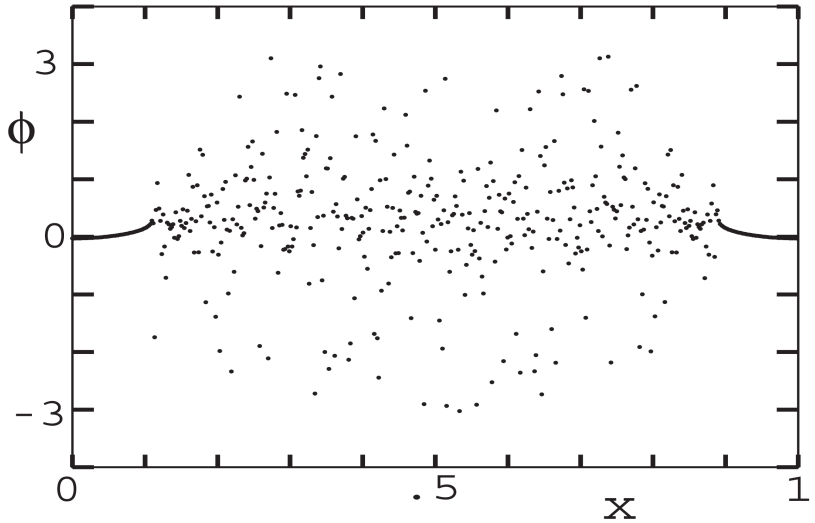
[Temirbayev et al, PRE, 2013]

Linear coupling Nonlinear coupling



Chimera states

Y. Kuramoto and D. Battogtokh observed in 2002 a symmetry breaking in non-locally coupled oscillators
$H(x) = \int dx' \exp[x' - x] Z(x')$



This regime was called "chimera" by Abrams and Strogatz

Chimera state as a pattern formation problem (with L. Smirnov, G. Osipov)

Start with equations for the phases:

$$\partial_t \phi = \omega + \mathrm{Im}\left[\exp\big(-\mathrm{i}\phi\,(x,t) - \mathrm{i}\alpha\big)\int G\,(x-\tilde{x})\exp\big(\mathrm{i}\phi\,(\tilde{x},t)\big)\mathrm{d}\tilde{x}\right],$$

$$G\,(y) = \kappa\exp\big(-\kappa\,|y|\big)/2$$

Introduce coarse-grained complex order parameter $Z\,(x,t) = \frac{1}{2\delta}\int_{x-\delta}^{x+\delta}\exp\big[\mathrm{i}\phi\,(\tilde{x},t)\big]\mathrm{d}\tilde{x}$ and reduce to a set of OA equations

$$\partial_t Z = i\omega Z + \left(\mathrm{e}^{-\mathrm{i}\alpha}H - \mathrm{e}^{\mathrm{i}\alpha}H^* Z^2\right)/2\,.$$

$$H\,(x,t) = \int G\,(x-\tilde{x})\,Z\,(\tilde{x},t)\,\mathrm{d}\tilde{x} \quad \Leftrightarrow \quad \partial_{xx}^2 H - \kappa^2 H = -\kappa^2 Z$$

System of partial differential equations can be analysed by standard methods

Chimera in two subpopulations

Model by Abrams et al:

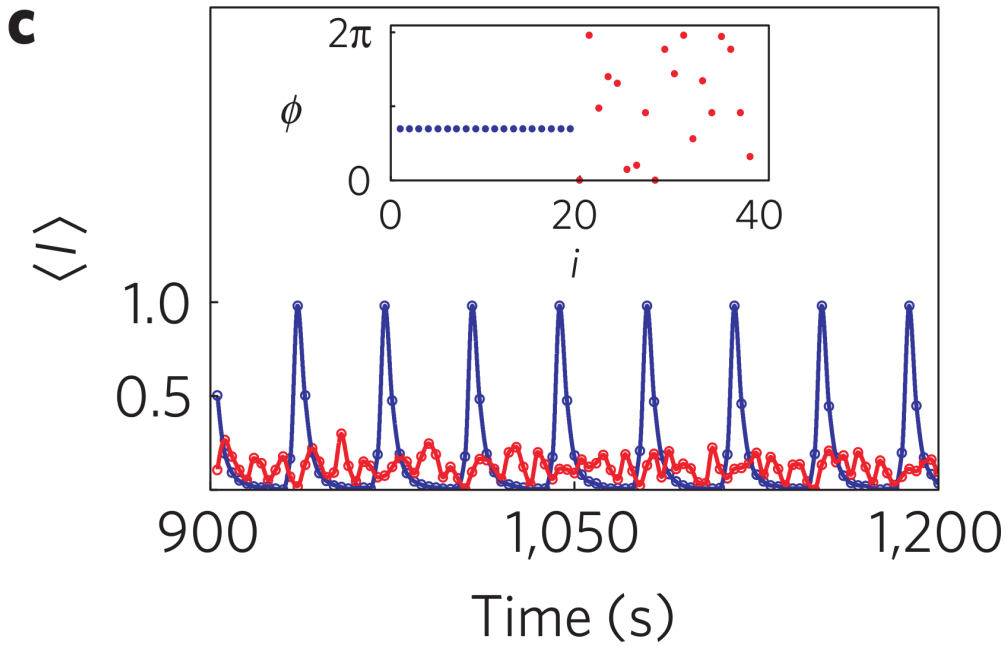
$$\dot{\varphi}_k^a = \omega + \mu \frac{1}{N} \sum_{j=1}^{N} \sin(\varphi_j^a - \varphi_k^a + \alpha) + (1-\mu) \sum_{j=1}^{N} \sin(\varphi_j^b - \varphi_k^a + \alpha)$$

$$\dot{\varphi}_k^b = \omega + \mu \frac{1}{N} \sum_{j=1}^{N} \sin(\varphi_j^b - \varphi_k^b + \alpha) + (1-\mu) \sum_{j=1}^{N} \sin(\varphi_j^a - \varphi_k^b + \alpha)$$

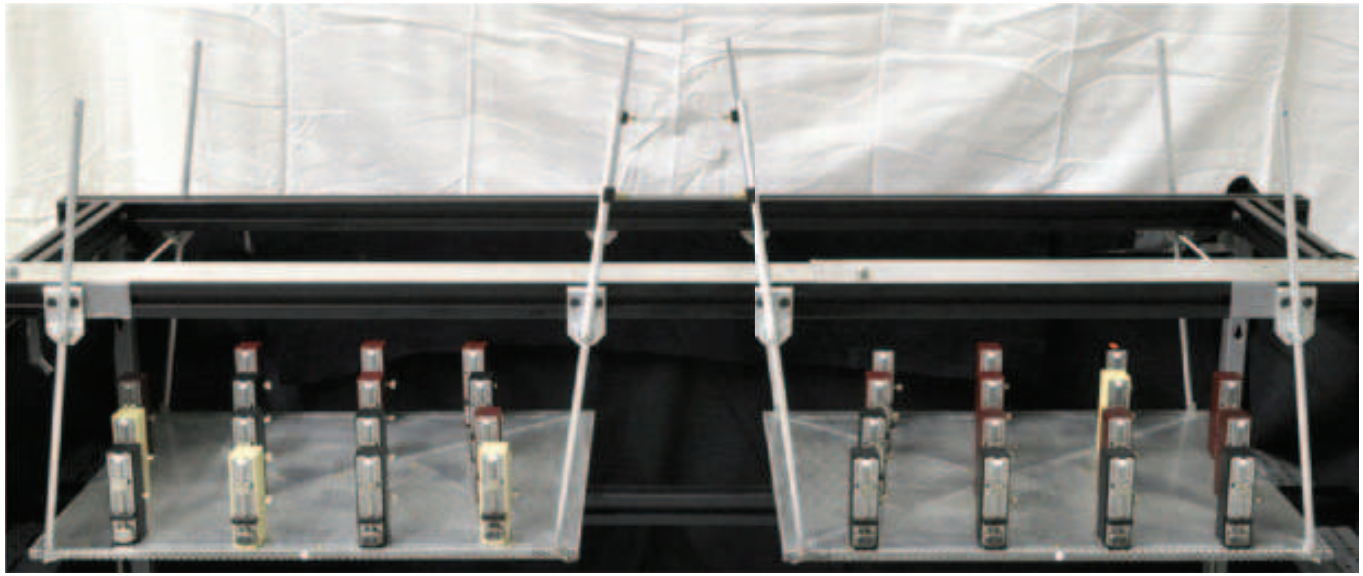
Two coupled sets of WS/OA equations: $\rho^a = 1$ and $\rho^b(t)$ quasiperiodic are observed

Chimera in experiments I

Tinsley et al: two populations of chemical oscillators



Chimera in experiments II



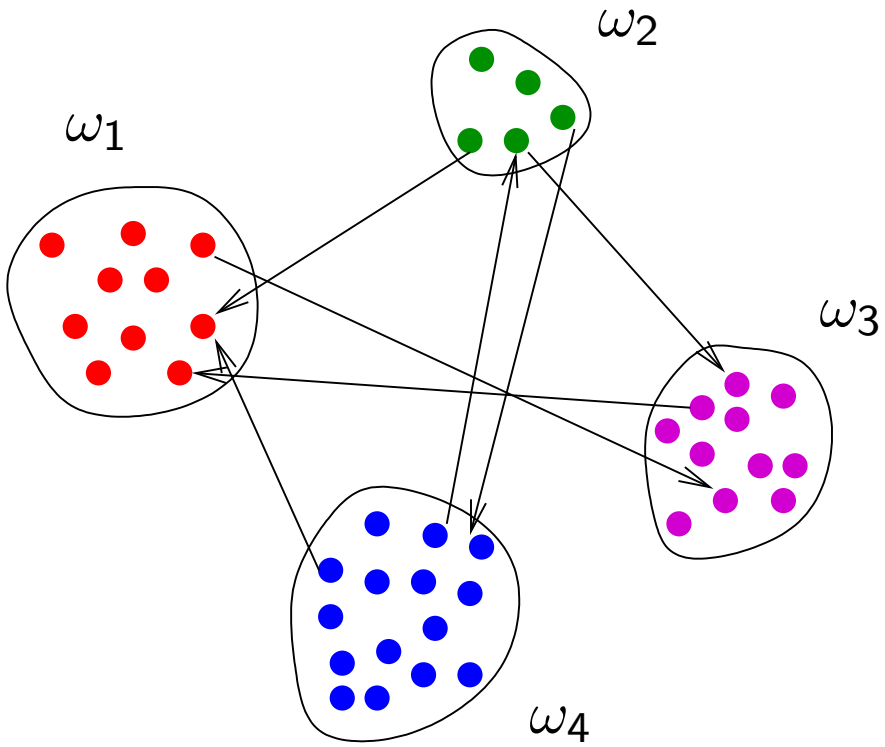
Erik A. Martens,
MPI für Dynamik und Selbstorganisation

Sets of oscillator populations

One population of nearly identical phase oscillators is described by WS/OA equations $\Rightarrow$ effective collective oscillator, complex amplitude = complex order parameter $0 \leq |Z| \leq 1$

Several such populations $\Rightarrow$ system of coupled "oscillators"

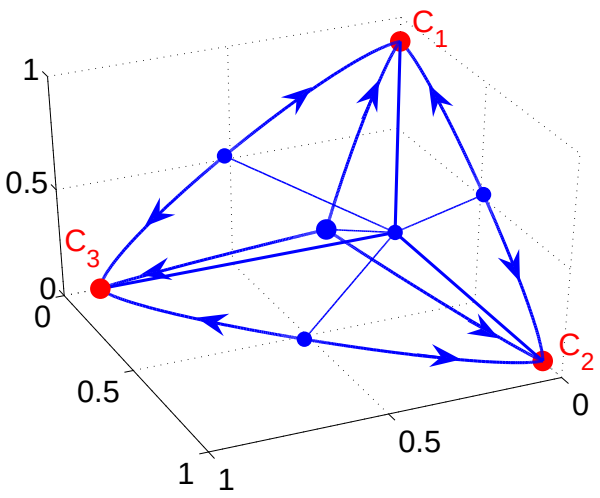
Non-resonantly interacting ensembles (with M. Komarov)



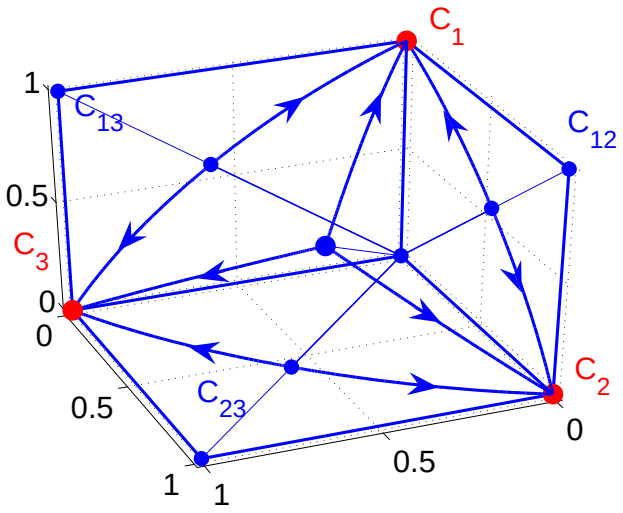
Frequencies are different – all interactions are non-resonant (only amplitudes of the order parameters involved)

$$\dot{\rho}_l = (-\Delta_l - \Gamma_{lm}\rho_m^2)\rho_l + (a_l + A_{lm}\rho_m^2)(1 - \rho_l^2)\rho_l, \qquad l = 1, \ldots, L$$

Competition for synchrony



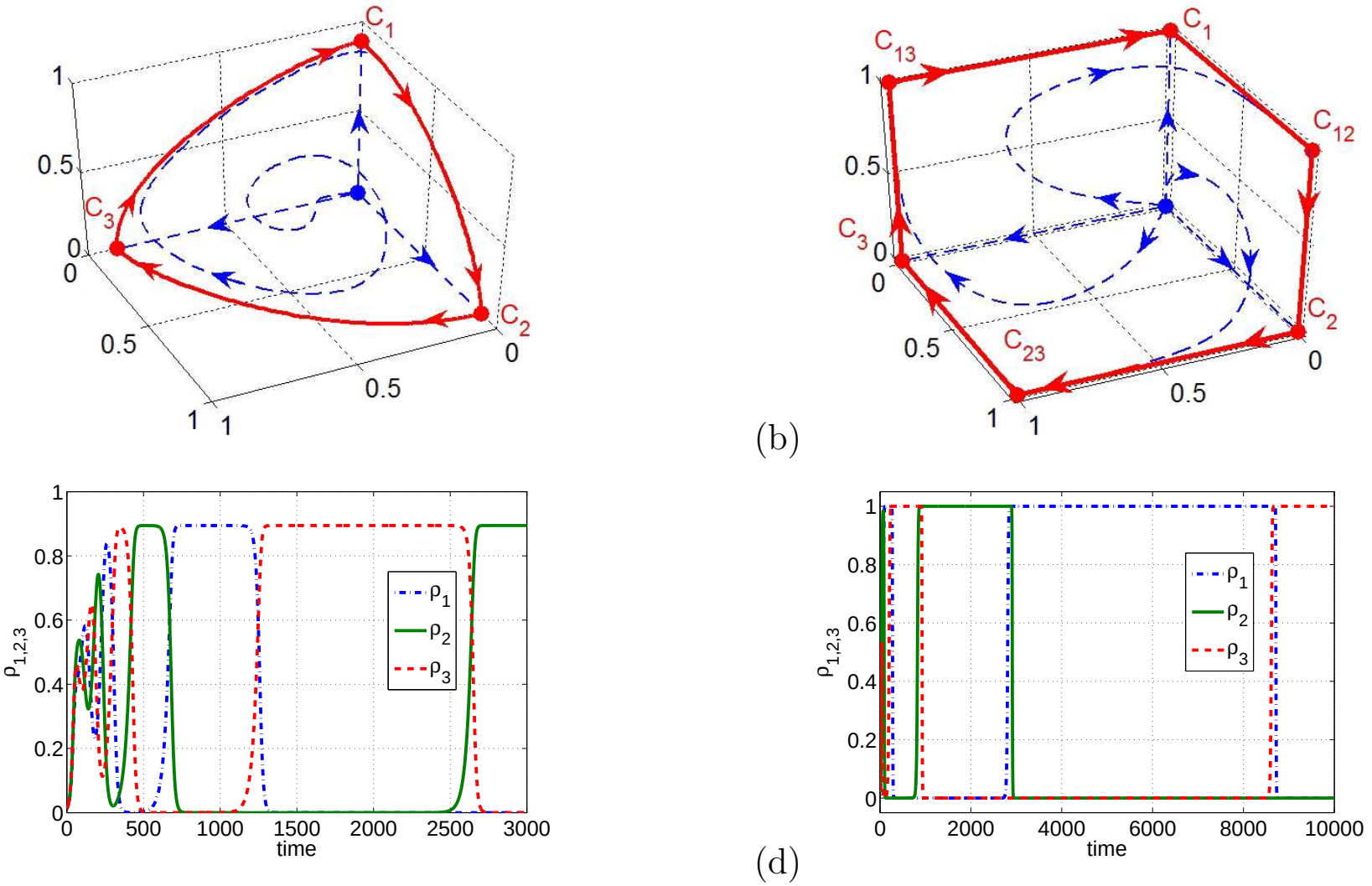
(a)



(b)

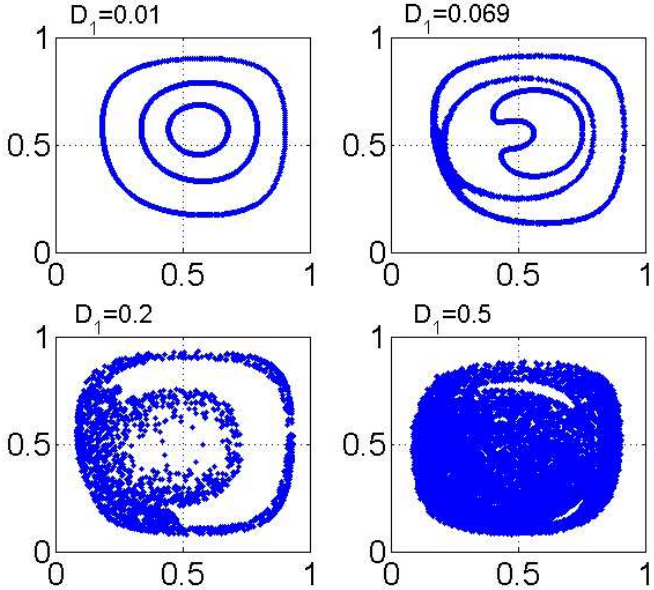
Only one ensemble is synchronous – depending on initial conditions

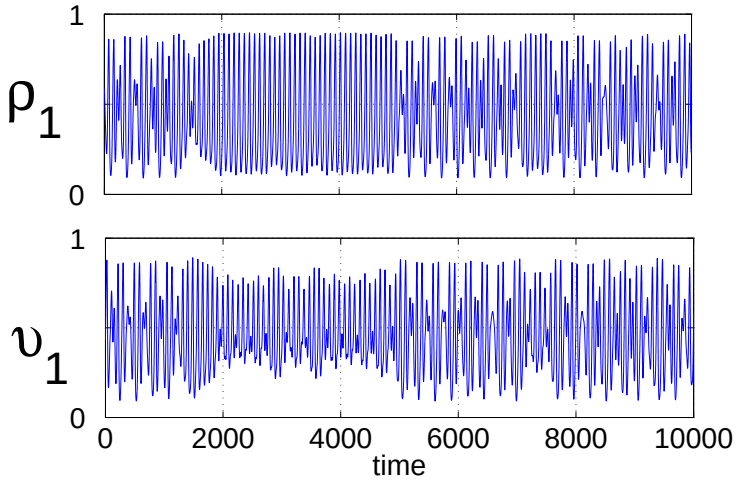
Heteroclinic synchrony cycles



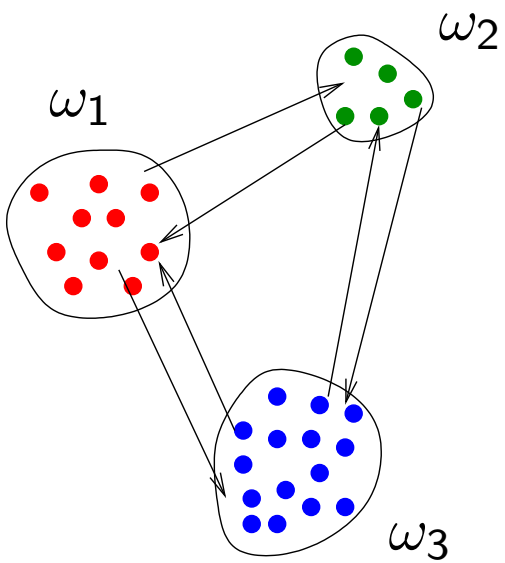
Chaotic synchrony cycles

Order parameters demonstrate chaotic oscillations





Resonantly interacting ensembles (with M. Komarov)



Most elementary nontrivial resonance $\omega_1 + \omega_2 = \omega_3$
Triple interactions:

$$\dot{\phi}_k = \ldots + \Gamma_1 \sum_{m,l} \sin(\theta_m - \psi_l - \phi_k + \beta_1)$$

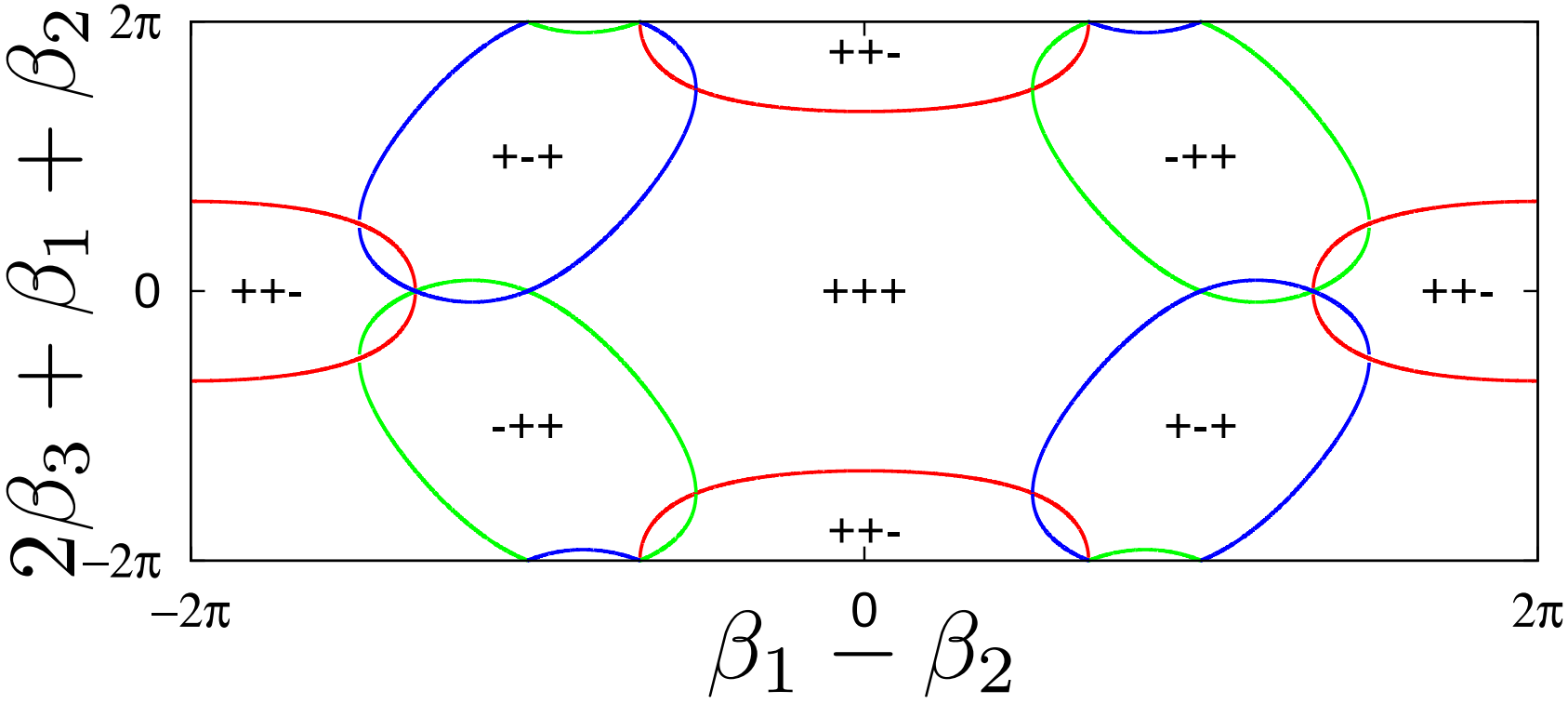
$$\dot{\psi}_k = \ldots + \Gamma_2 \sum_{m,l} \sin(\theta_m - \phi_l - \psi_k + \beta_2)$$

$$\dot{\theta}_k = \ldots + \Gamma_3 \sum_{m,l} \sin(\phi_m + \psi_l - \theta_k + \beta_3)$$

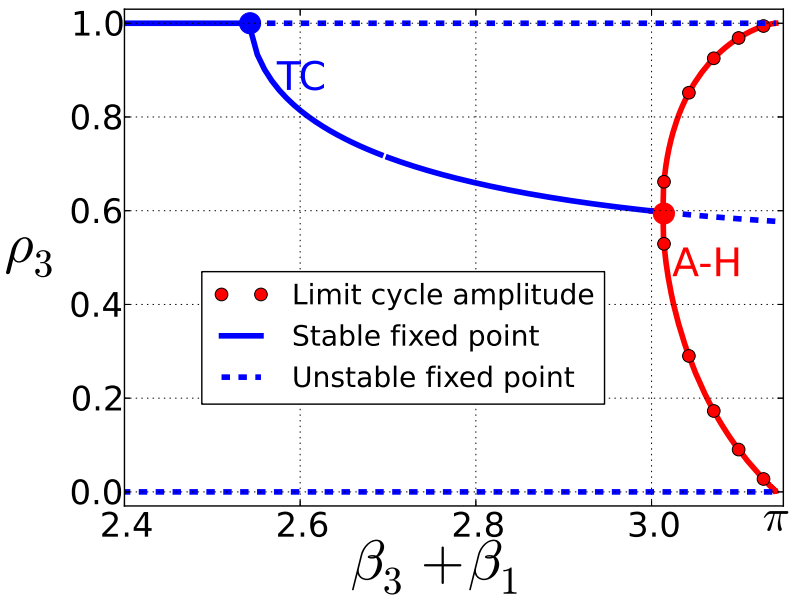
Set of three OA equations

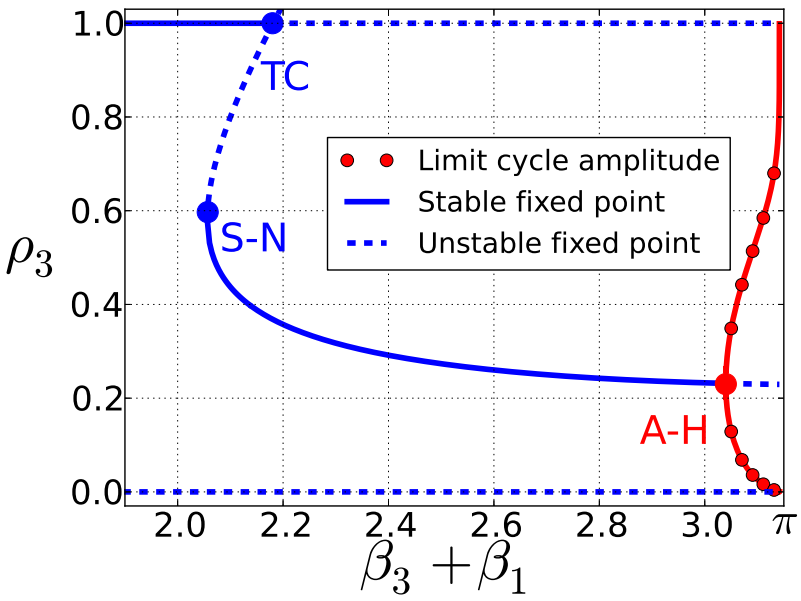
$$\dot{z}_1 = z_1(i\omega_1 - \delta_1) + (\epsilon_1 z_1 + \gamma_1 z_2^* z_3 - z_1^2(\epsilon_1^* z_1^* + \gamma_1^* z_2 z_3^*))$$
$$\dot{z}_2 = z_2(i\omega_2 - \delta_2) + (\epsilon_2 z_2 + \gamma_2 z_1^* z_3 - z_2^2(\epsilon_2^* z_2^* + \gamma_2^* z_1 z_3^*))$$
$$\dot{z}_3 = z_3(i\omega_3 - \delta_3) + (\epsilon_3 z_3 + \gamma_3 z_1 z_2 - z_3^2(\epsilon_3^* z_3^* + \gamma_3^* z_1^* z_2^*))$$

Regions of synchronizing and desynchronizing effect from triple coupling



Bifurcations in dependence on phase constants





Beyond WS and OA theory: bi-harmonic coupling (with M. Komarov)

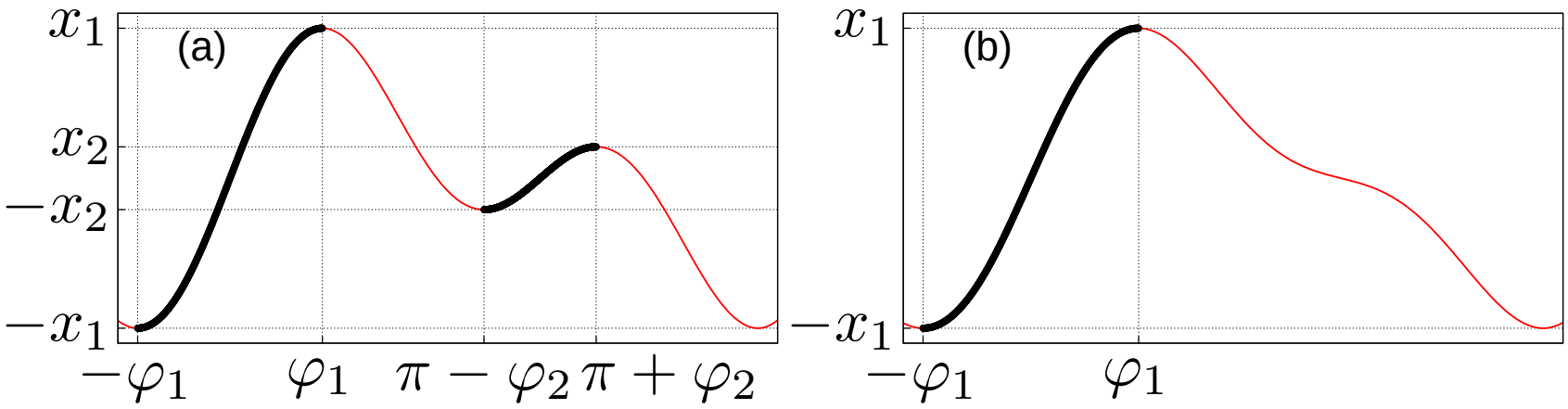
$$\dot{\varphi}_k = \omega_k + \frac{1}{N}\sum_{j=1}^{N}\Gamma(\phi_j - \phi_k) \qquad \Gamma(\psi) = \varepsilon\sin(\psi) + \gamma\sin(2\psi)$$

Corresponds to XY-model with nematic coupling

$$H = J_1\sum_{ij}\cos(\theta_i - \theta_j) + J_2\sum_{ij}\cos(2\theta_i - 2\theta_j)$$

Multi-branch entrainment

$$\dot{\varphi} = \omega - \varepsilon R_1 \sin(\varphi) - \gamma R_2 \sin(2\varphi)$$



Self-consistent theory in the thermodynamic limit

Two relevant order parameters $R_m e^{i\Theta_m} = N^{-1} \sum_k e^{im\phi_k}$ for $m = 1, 2$ Dynamics of oscillators (due to symmetry $\Theta_{1,2} = 0$)

$$\dot{\varphi} = \omega - \varepsilon R_1 \sin(\varphi) - \gamma R_2 \sin(2\varphi)$$

yields a stationary distribution function $\rho(\varphi|\omega)$ which allows one to calculate the order parameters

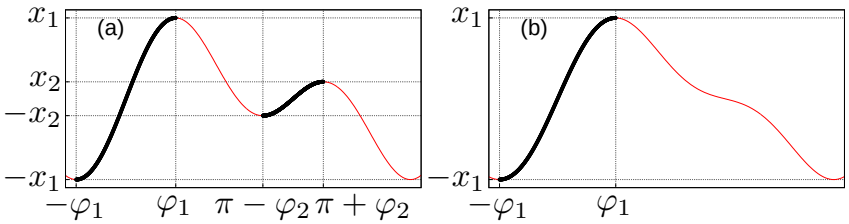
$$R_m = \iint d\varphi d\omega \; g(\omega)\rho(\varphi|\omega) \cos m\varphi, \qquad m = 1, 2$$

Where $g(\omega)$ is the distribution of natural frequencies

Multiplicity at multi-branch locking

Three shapes of phase distribution

$$\rho(\varphi|\omega) = \begin{cases} (1 - S(\omega))\delta(\varphi - \Phi_1(\omega)) + \\ \quad + S(\omega)\delta(\varphi - \Phi_2(\omega)) & \text{for two branches} \\ \delta(\varphi - \Phi_1(\omega)) & \text{for one locked bracnch} \\ \frac{C}{|\dot{\varphi}|} & \text{for non-locked} \end{cases}$$



$0 \leq S(\omega) \leq 1$ is an **arbitrary** indicator function

Explicit (parametric) solution of the self-consistent eqs

We introduce

$$\cos\theta = \gamma R_2/R, \quad \sin\theta = \varepsilon R_1/R, \quad R = \sqrt{\gamma^2 R_2^2 + \varepsilon^2 R_1^2}, \quad x = \omega/R$$

so that the equation for the locked phases is

$$x = y(\theta,\varphi) = \sin\theta\sin\varphi + \cos\theta\sin 2\varphi$$

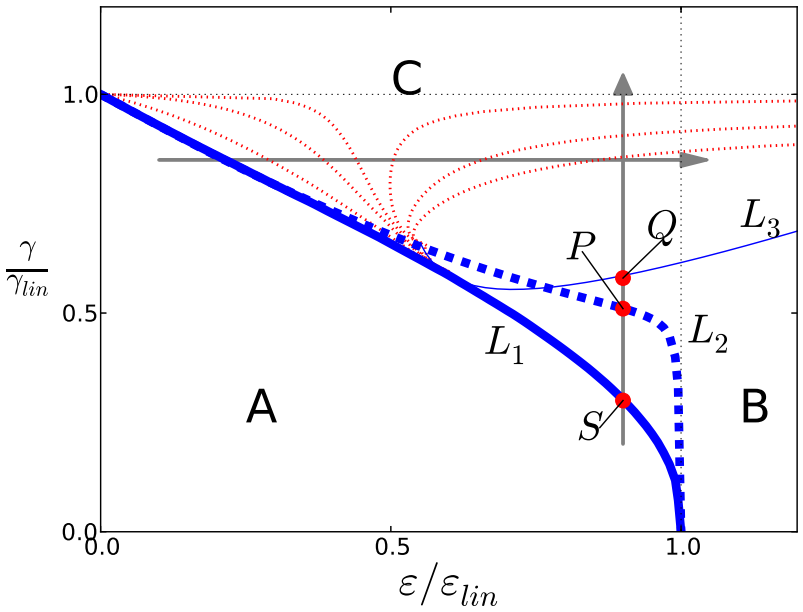
Then by calculating two integrals

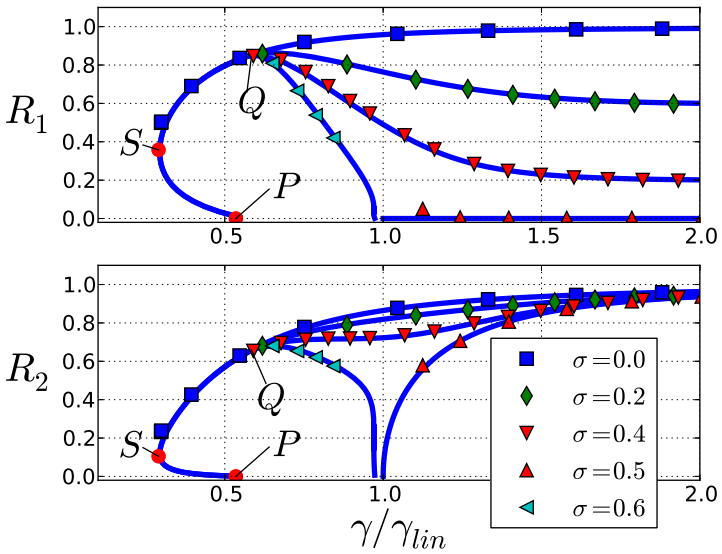
$$F_m(R,\theta) = \int_{-\pi}^{\pi} d\varphi \cos m\varphi \left[A(\varphi) g(Ry) \frac{\partial y}{\partial \varphi} + \int_{|x|>x_1} dx \, \frac{C(x,\theta)}{|x - y(\theta,\varphi)|} \right]$$

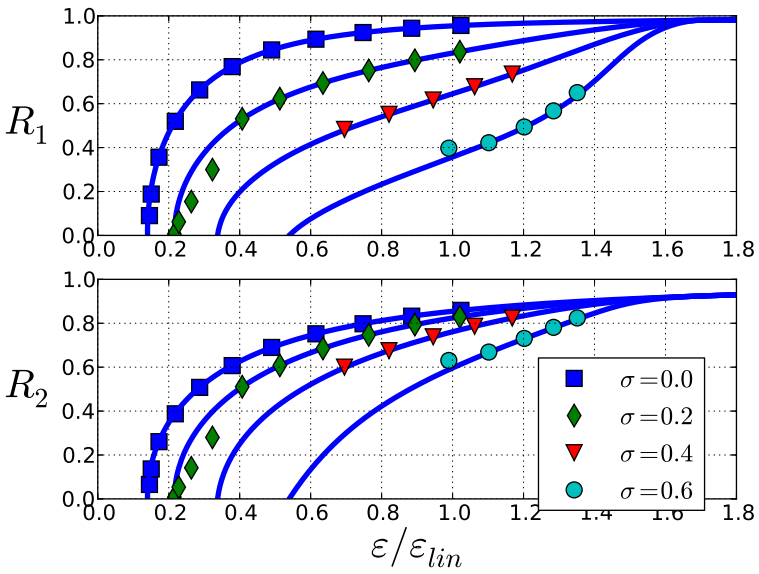
we obtain a solution

$$R_{1,2} = R F_{1,2}(R,\theta), \quad \varepsilon = \frac{\sin\theta}{F_1(R,\theta)}, \quad \gamma = \frac{\cos\theta}{F_2(R,\theta)}$$

Phase diagram of solutions



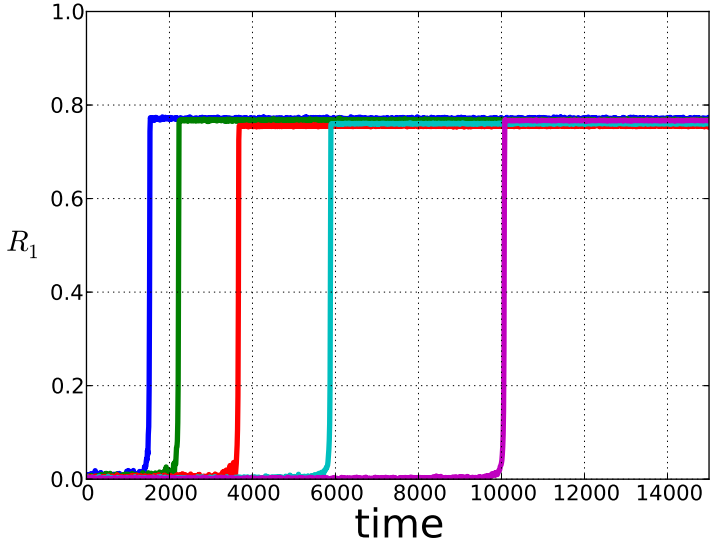


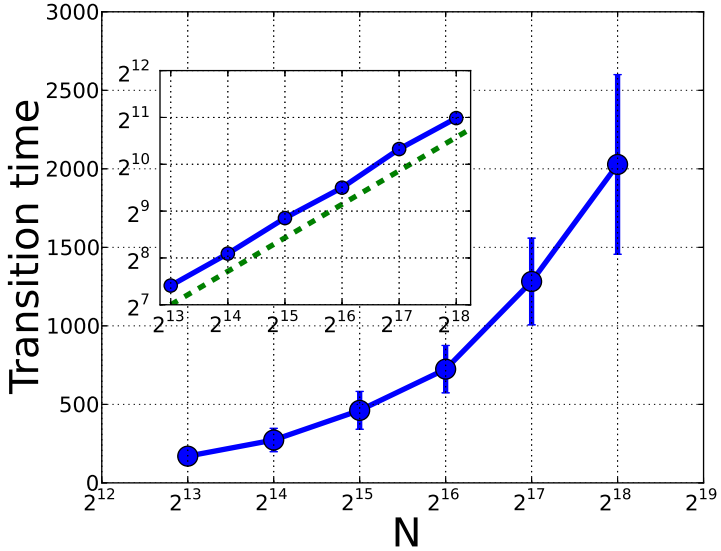


Stability issues

We cannot analyze stability of the solutions analytically (due to signularity of the states), but can perform simulations of finite ensembles

Nontrivial solution coexist with netrally stable asynchronous state





$N = 5 \cdot 10^4, 10^5, 2 \cdot 10^5, 5 \cdot 10^5, 10^6$

$T \propto N^{0.72}$

Perturbation theory for WS integrability (WIth V. Vlasov and M. Rosenblum)

$$\dot{\varphi}_k = \omega(t) + \mathrm{Im}\left[H(t)e^{-\mathrm{i}\varphi_k}\right] + F_k \ , \quad k = 1, \ldots, N \ .$$

We seek for a WS equation with a correction term P

$$\dot{z} = \mathrm{i}\omega z + \frac{H}{2} - \frac{H^*}{2}z^2 + P \ .$$

Evolution of constants:

$$\dot{\psi}_k = \omega + \mathrm{Im}(z^* H) + F_k \left[\frac{2\mathrm{Re}\left(ze^{-\mathrm{i}\psi_k}\right) + 1 + |z|^2}{1 - |z|^2}\right] - \frac{2\mathrm{Im}\left[P\left(z^* + e^{-\mathrm{i}\psi_k}\right)\right]}{1 - |z|^2} \ .$$

Expression for correction in the WS equation:

$$P = \frac{\mathrm{i}}{(1-|U|^2)N} \sum_{k=1}^{N} F_k \Big[z(1 - Ue^{-2\mathrm{i}\psi_k}) + \\ + (1+|z|^2)(e^{\mathrm{i}\psi_k} - Ue^{-\mathrm{i}\psi_k}) + z^*(e^{2\mathrm{i}\psi_k} - U) \Big] \ .$$

where $U = N^{-1} \sum_k \exp[\mathrm{i}2\psi_k]$

Perturbation results in simplest cases

Small nonidentity $\sim \varepsilon$ of oscillators (but still sine-coupling) or small white noise with intensity $\sim \varepsilon^2$:

Corrections $\sim \varepsilon^2$ to the WS equation; the distribution of constants deviates from the uniform one also as $\sim \varepsilon^2$
For example, for noise

$$w(\theta) \approx (2\pi)^{-1} \left(1 - \varepsilon^2 \frac{2\rho^2}{(1-\rho^2)^2\Omega} \sin 2\theta\right)$$

$$P = -\varepsilon^2 \frac{2z(1+|z|^2)}{1-|z|^2}$$

Conclusions

- Closed equations for the order parameters evolution (Watanabe-Strogatz variables for identical and Ott-Antonsen eqs for certain non-identical)
- Transition to partial synchronization (self-organized quasiperiodicity)
- Many populations can be treated as effective coupled oscillators, for which complex amplitude = complex order parameter
- Chimera states as patterns in PDEs for complex order parameter
- Resonantly and nonresonantly interacting populations
- Bi-harmonic coupling: multi-branch entrainment
- Beyond validity of WS/OA approaches a perturbation method has been constructed