Hidden nonlinear oscillations in dynamical systems

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Content

Attractors in dynamical systems

Attractors is bounded, closed, invariant, and attracting (in some sense) set.

$\dot{u} = f(u), \quad u \in \mathbb{R}^n, \; f : \mathbb{R}^n \to \mathbb{R}^n$ solution $u(t, u_0)$ such that $u(0, u_0) = u_0$.

(i) *(local) attractor* is a locally attractive set (for a "neighborhood" of the set)

(ii) *global attractor* is a globally attractive set (for the "whole" phase space)

It is reasonable to consider only minimal global and local attractors, i.e. the smallest bounded closed invariant set possessing the property (ii) or (i).
Various types of attraction: Milnor attractor, B-attractor and others

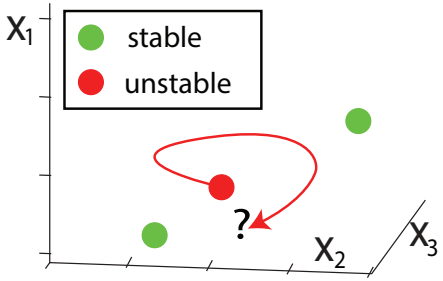
An oscillation can be visualized numerically if initial data from its vicinity lead to a long-term behavior that approaches the oscillation. From a computational point of view, such an oscillation (or a set osc.) is called an attractor, and its attracting set is called the basin of attraction. How to choose initial data in the basin of attraction?

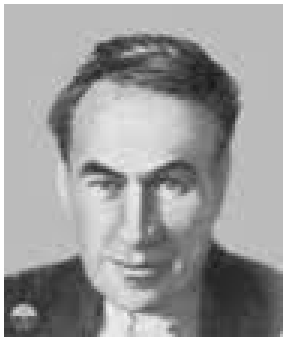
Sustained chaos vs Transient chaos
Self-excited attractor vs Hidden attractor

Numerical localization of attractors

An oscillation can be visualized numerically if initial data from its vicinity lead to a long-term behavior that approaches the oscillation. From a computational point of view, such an oscillation (or a set osc.) is called an attractor, and its attracting set is called the basin of attraction. How to choose initial data in the basin of attraction?

$$\dot{\mathbf{x}} = \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dots \\ \dot{x}_n \end{pmatrix} = \begin{pmatrix} f_1(x_1, x_2, .., x_n) \\ f_2(x_1, x_2, .., x_n) \\ \dots \\ f_n(x_1, x_2, .., x_n) \end{pmatrix} = \mathbf{F}(\mathbf{x}) \quad \mathbf{x}_{t+1} = \mathbf{F}(\mathbf{x}_t)$$

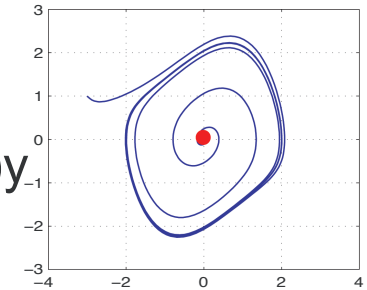




H. Barkhausen (192x) selbsterregten Schwingungen;
A. Andronov (1929) self (excited) **oscillations** : the *generation* and *maintenance* of a periodic motion by a source of power that lacks any corresponding periodicity. [*transition process* to the *limit cycle* in Van der Pol model]

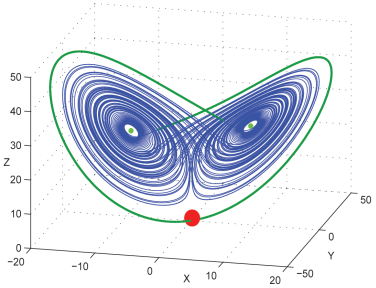
Van der Pol

$\dot{x} = y$
$\dot{y} = -x + e(1-x^2)y$



Lorenz

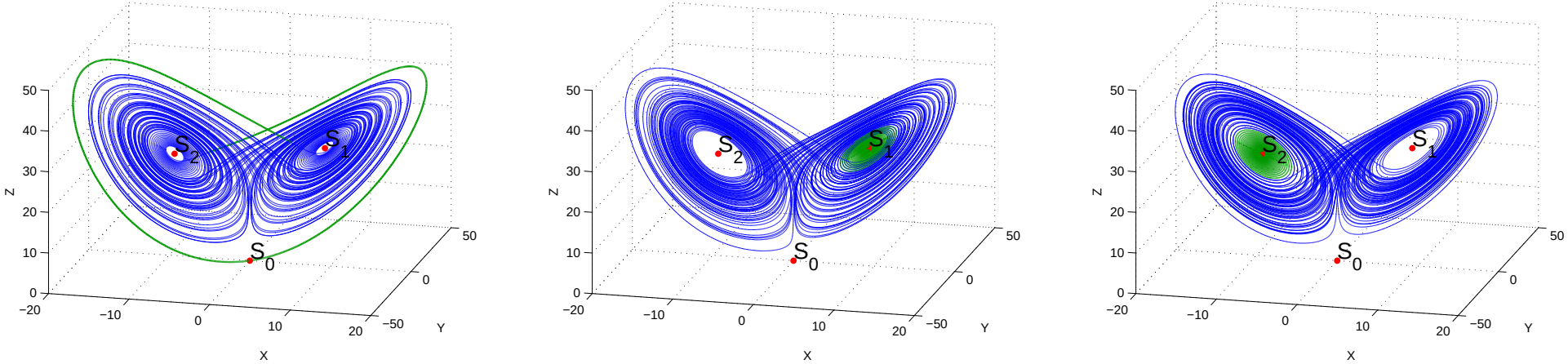
$\dot{x} = -s(x - y)$
$\dot{y} = rx - y - xz$
$\dot{z} = -bz + xy$



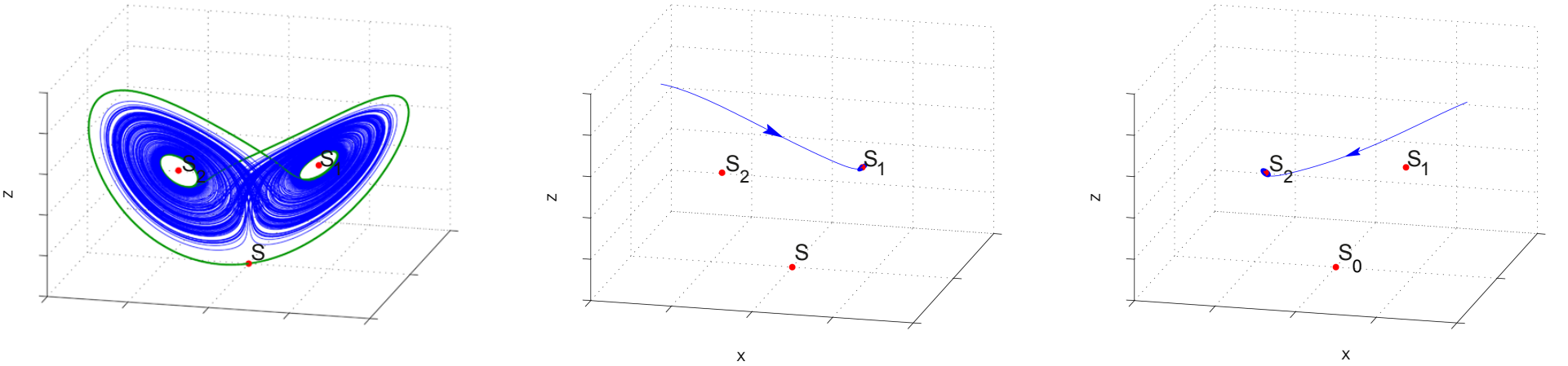
If attractors's basin of attraction doesn't overlap with vicinities of equilibria?

Visualization of the Lorenz attractor (1963) and multistability

$r=28,\ \sigma=10,\ b=8/3$: self-excited with respect to $S_0, S_\pm$, monostability



$r=24.5,\ \sigma=10,\ b=8/3$: self-excited with respect to S_0, multistability

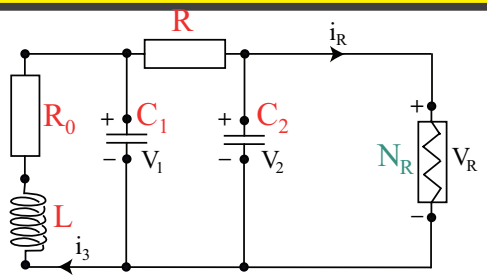


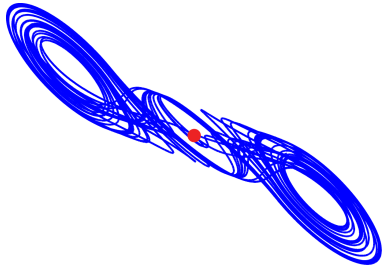
Existence of the Lorenz attractor which is not connected with equilibria?

Self-excited and hidden attractors: Chua circuit

Leonov-Kuznetsov, 2009: An attractor is called a self-excited attractor if its basin of attraction intersects with small neighborhood of an equilibrium, otherwise it is called a hidden attractor. E.g. hidden attractors are attractors in the systems without equilibria, or with the only stable equilibrium (a case of multistability).

$$\dot{x} = \alpha\big(y - x - f(x)\big),$$
$$\dot{y} = x - y + z,$$
$$\dot{z} = -(\beta y + \gamma z),$$

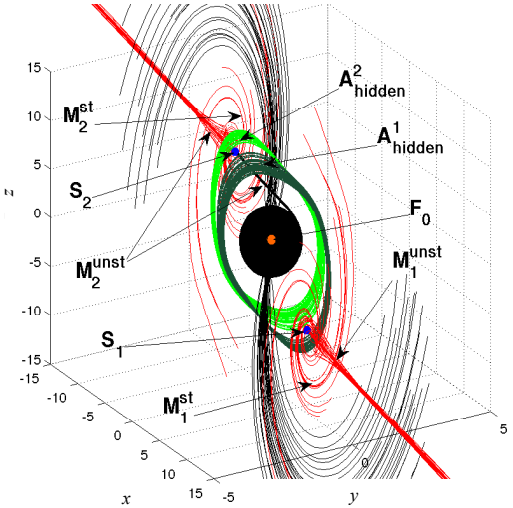




$$f(x) = m_1 x + \mathrm{sat}(x) = m_1 x + \tfrac{1}{2}(m_0 - m_1)(|x+1| - |x-1|)$$

L.Chua, 1990 (**conjecture**): If the zero equilibrium is stable $\Rightarrow$ no attractors.

Two symmetric hidden chaotic attractors ($\mathcal{A}^{1,2}_{\text{hidden}}$ - green domains) in the classical Chua system. trajectories (red) from unstable manifolds $\mathrm{M}^{\text{unst}}_{1,2}$ of two saddle points $S_{1,2}$ are either attracted to the locally stable zero equilibrium F_0, or tend to infinity; trajectories (black) from stable manifolds $\mathrm{M}^{\text{st}}_{0,\pm}$ tend to F_0 or $S_{1,2}$



✓ Leonov G.A.,Kuznetsov N.V., Vagaitsev V.I, Localization of hidden Chua's attractors, Physics Letters A, 375(23), 2011, 2230-2233

Criteria of periodic solution existence (DFM)

$$\dot{\mathbf{x}}=\mathbf{P}\mathbf{x}+\mathbf{q}\psi(\mathbf{r}^*\mathbf{x}) : W(p)=\mathbf{r}^*(\mathbf{P}-p\mathbf{I})^{-1}\mathbf{q}, \operatorname{Im}W(i\omega)=0, k=-(\operatorname{Re}W(i\omega))^{-1}$$
$$\dot{\mathbf{x}}=\mathbf{P}_0\mathbf{x}+\mathbf{q}\varphi(\mathbf{r}^*\mathbf{x}) : \mathbf{P}_0=\mathbf{P}+\mathbf{k}\mathbf{q}\mathbf{r}^*, \lambda_{1,2}^{\mathbf{P}_0}=\pm i\omega, \operatorname{Re}\lambda_{j>2}^{\mathbf{P}_0}<0, \varphi=\psi-k\mathbf{r}^*\mathbf{x}$$
$$\mathbf{x}=\mathbf{S}\mathbf{y}, \mathbf{A}=\mathbf{S}^{-1}\mathbf{P_0}\mathbf{S}, \mathbf{b}=(b_1,b_2,\mathbf{b}_3)^*=\mathbf{S}^{-1}\mathbf{q}, \mathbf{c}^*=(1,0,\mathbf{c}_3)^*=\mathbf{r}^*\mathbf{S}$$

Harmonic linearization, linear transformation, small parameter method

$$\begin{aligned}\dot{y}_1&=-\omega y_2+b_1\varepsilon\varphi(y_1+\mathbf{c}_3^*\mathbf{y}_3)\\ \dot{y}_2&=\omega y_1+b_2\varepsilon\varphi(y_1+\mathbf{c}_3^*\mathbf{y}_3)\\ \dot{\mathbf{y}}_3&=\mathbf{A}_3\mathbf{y}_3+\mathbf{b}_3\varepsilon\varphi(y_1+\mathbf{c}_3^*\mathbf{y}_3)\\ \mathbf{y}_3^*(\mathbf{A}_3+\mathbf{A}_3^*)\mathbf{y}_3&\le -2d|\mathbf{y}_3|^2\end{aligned}$$

$$\begin{aligned}y_1(t)&=\cos(\omega t)y_1(0)+O(\varepsilon)\\ y_2(t)&=\sin(\omega t)y_1(0)+O(\varepsilon)\\ \mathbf{y}_3(t)&=e^{\mathbf{A}_3 t}\mathbf{y}_3(0)+\mathbf{O_{n-2}}(\varepsilon)\\ t&\in(0,T]\end{aligned}$$

$$\mathbf{y}(0)\in\Omega=\{y_1\in[a_1,a_2], y_2=0, |\mathbf{y}_3|\le D\varepsilon\}, \mathbf{F}\mathbf{y}(0)=\mathbf{y}(T), T=\frac{2\pi}{\omega}+O(\varepsilon)$$

Theorem. If there exists $a_0>0$: $\Phi(a_0)=\int\limits_0^{2\pi/\omega}\varphi\big(\cos(\omega t)a_0\big)\cos(\omega t)dt=0$ and $b_1\frac{d\Phi(a)}{da}\Big|_{a=a_0}<0$ then exists periodic solution $\mathbf{x}(t)=\mathbf{S}\mathbf{y}(t)$ with initial data $\mathbf{x}(0)=\mathbf{S}\,(a_0+O(\varepsilon),\,0,\,\mathbf{O_{n-2}}(\varepsilon))^*$.

Hidden oscillation localization: analytical-numerical procedure

(1) $\dot{\mathbf{x}}=\mathbf{F}(\mathbf{x})$ (2) $\dot{\mathbf{x}}=\mathbf{F}_{\lambda=a}(\mathbf{x})$ (3) $\dot{\mathbf{x}}=F_{\lambda_j\in(a,b]}(\mathbf{x}),\ \ \mathbf{F}_{\lambda=b}(\mathbf{x})\equiv\mathbf{F}(\mathbf{x})$

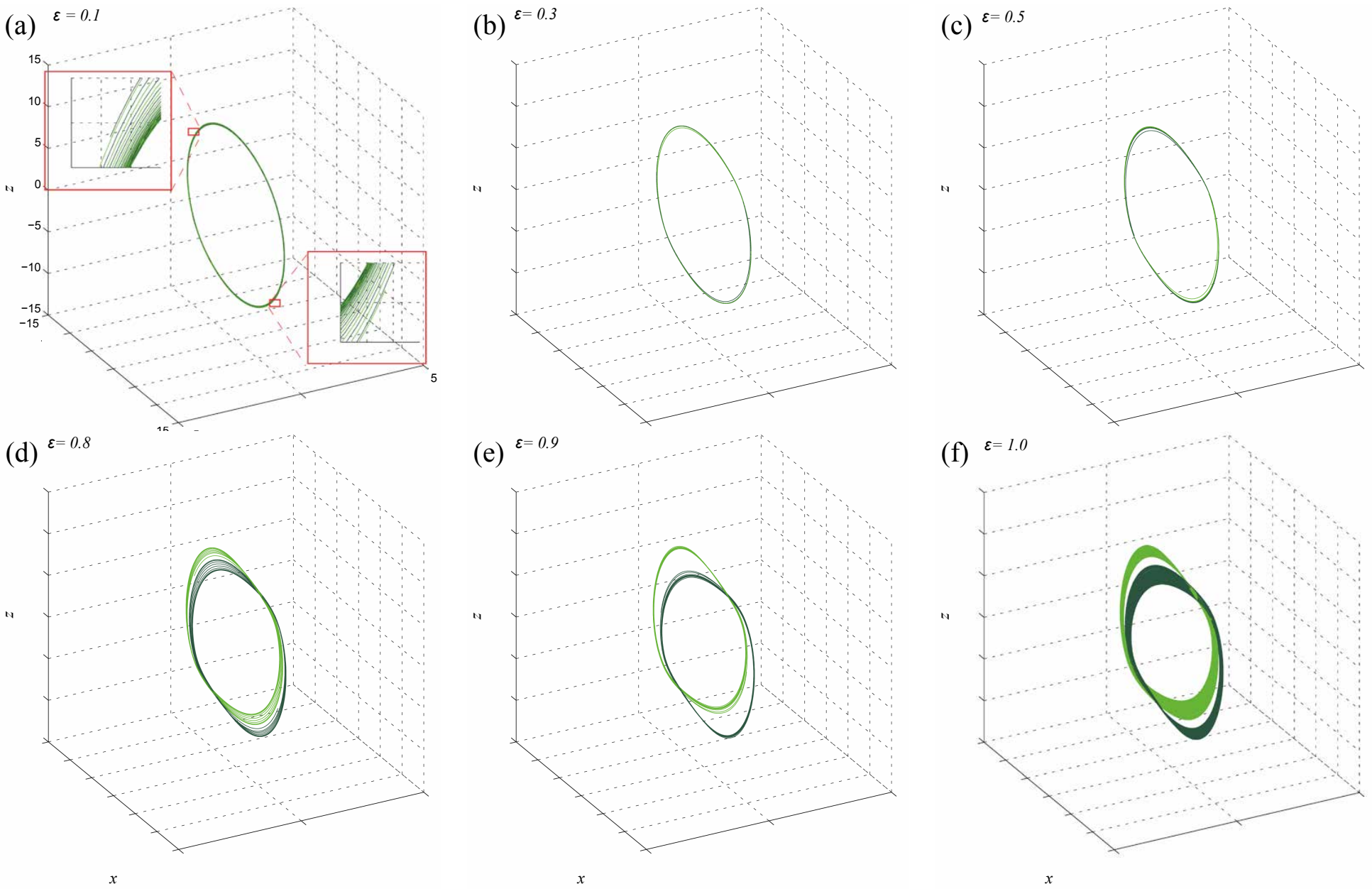
In the procedure, computational methods and the engineering notion of a **transient process** are combined to study oscillations:
1) Choose or add a parameter λ in such a way that for $\lambda = a$ analytically or computationally one can find a nontrivial attractor $\mathcal{A}_0$ in (2)
(often this attractor has a simple form, e.g.: periodic, self-excited)
2) Localize an attractor $\mathcal{A}$ in (1): numerically follow transformation of $\mathcal{A}_j$ with increasing $j=0$: $\lambda_0=a, ..., j=m$: $\lambda_m=b$ ($\mathcal{A}=\mathcal{A}_m$ in (3))

1. If all points of $\mathcal{A}_0$ are in the basin of attraction of $\mathcal{A}_1$ (oscillating attractor of (3) with $j=1$), then a trajectory $\mathbf{x}^1(t)$ with an initial data $\mathbf{x}^1(0)=\mathbf{x}^0(0)\in\mathcal{A}_0$ is attracted and identified $\mathcal{A}_1$ of (3) with $j=1$.

2. Numerically investigate changes to the shape of $\mathcal{A}_1$. If the change in λ (from λ_j to λ_{j+1} in (3)) does not cause a loss of the stability bifurcation of $\mathcal{A}_j$: i.e. if, in the computation, the trajectory $\mathbf{x}^{j+1}(t)$ is not attracted to equilibria or ∞ (for suff. large time interval $[0,T]$), then $\mathbf{x}^{j+1}(t)$ identify an attractor $\mathcal{A}_{j+1}$, and one can start trajectory $\mathbf{x}^{j+2}(t)$ with $\mathbf{x}^{j+2}(0)=\mathbf{x}^{j+1}(T)$ (last point on previous step).

✓ Leonov G.A., Kuznetsov N.V., Mokaev T.N., Homoclinic orbits, and self-excited and hidden attractors in a Lorenz-like system describing convective fluid motion, **European Physical Journal Special Topics**, 224, **2015**, 1421-1458

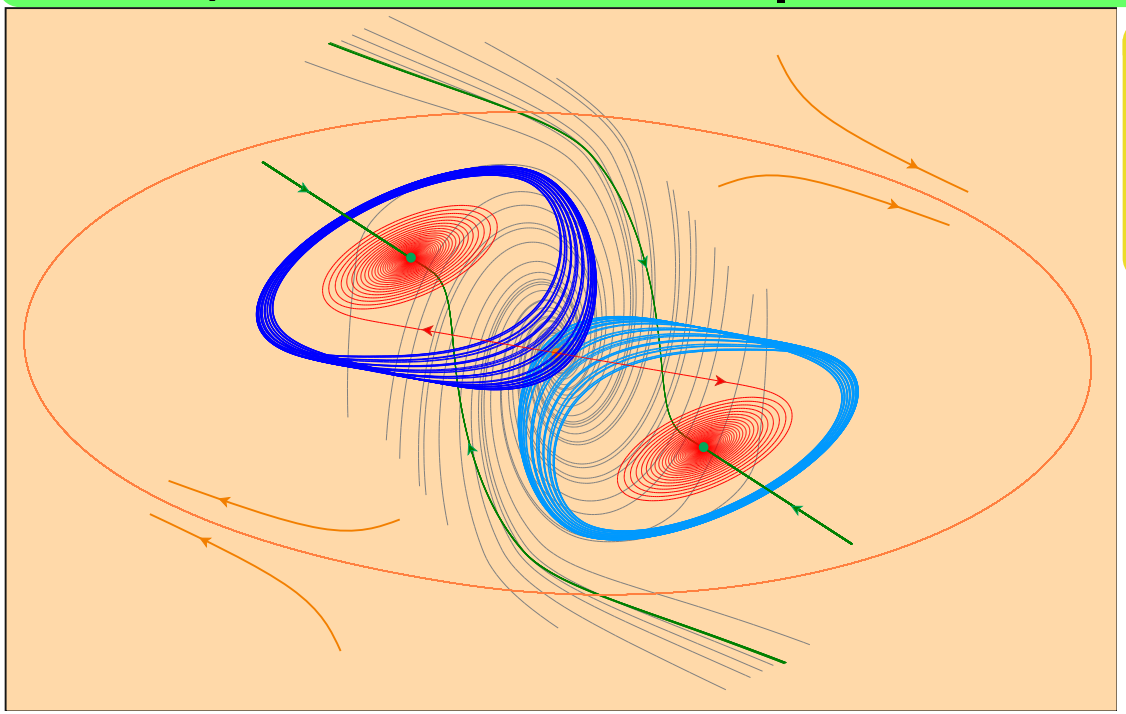
Scenario of transition to chaos in Chua circuit:



Hidden Chua attractors and multistability: 5 coexisting attractors

"Chaotic" generalization of Hilbert's 16th problem: on **the number and disposition of attractors** in multi-dimensional chaotic dynamical systems; e.g., depending on the degree of polynomials in the rhs of the model.

[Leonov G., Kuznetsov N., On differences and similarities in the analysis of Lorenz, Chen and Lu systems, **Applied Mathematics and Computation**, 256, **2015**, 334-343]



What is the maximum number of coexisting attractors that can be found in the Chua system? How many of the coexisting attractors can be hidden?

2 trivial attractors (stable equilibria – green points) unstable equilibrium (red point) (grey – 2d stable manifold, red – 1d unstable manifold attracted by 2 trivial attractors);

3 hidden Chua attractors: periodic limit cycle (orange) and two symmetric chaotic Chua attractors (blue)

✓ Kuznetsov N., Kuznetsova O., Mokaev T., Leonov G., Stankevich N., Hidden attractors localization in Chua's circuit via the describing function method, **IFAC-PapersOnLine**, 50(1), **2017**, 2651-2656

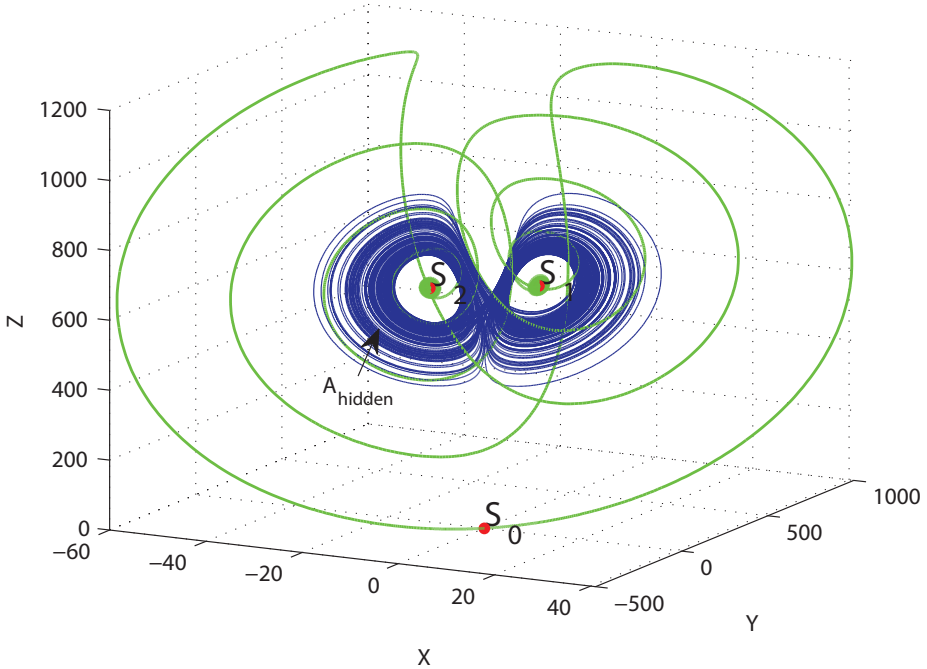
✓ N.V. Stankevich, N.V. Kuznetsov, G.A. Leonov, L. Chua, Scenario of the birth of hidden attractors in the Chua circuit, **Int. J. of Bifurcation and Chaos**, 27(12), **2017**, art. num. 1730038

Hidden attractor in a Lorenz-like system

Glukhovsky–Dolghansky (1980): fluid motion in an ellipsoidal rotating cavity can be interpreted as one of the models of ocean flows ($a > 0$)

$$\dot{x} = -\sigma(x - y) - ayz$$
$$\dot{y} = rx - y - xz$$
$$\dot{z} = -bz + xy$$
$$b = 1, r = 700, a = 0.0052, \sigma = 4$$

Equilibria: saddle S_0 & 2 stable $S_{1,2}$
Trajectories from S_0 tend (green) to zero $S_{1,2}$;
Hidden attractor (blue) coexist with B-attractor (green outgoing separatrices of the saddle S_0 attracted to the red equilibria $S_{1,2}$)



✓ Leonov G.A., Kuznetsov N.V., Mokaev T.N., Hidden attractor and homoclinic orbit in Lorenz-like system describing convective fluid motion in rotating cavity, **Commun Nonlinear Sci Numer Simulat**, 28(1-3), **2015**, 166-174 (doi: 10.1016/j.cnsns.2015.04.007)
✓ Leonov G.A., Kuznetsov N.V., Mokaev T.N., Homoclinic orbits, and self-excited and hidden attractors in a Lorenz-like system describing convective fluid motion, **European Physical Journal Special Topics**, 224, **2015**, 1421-1458

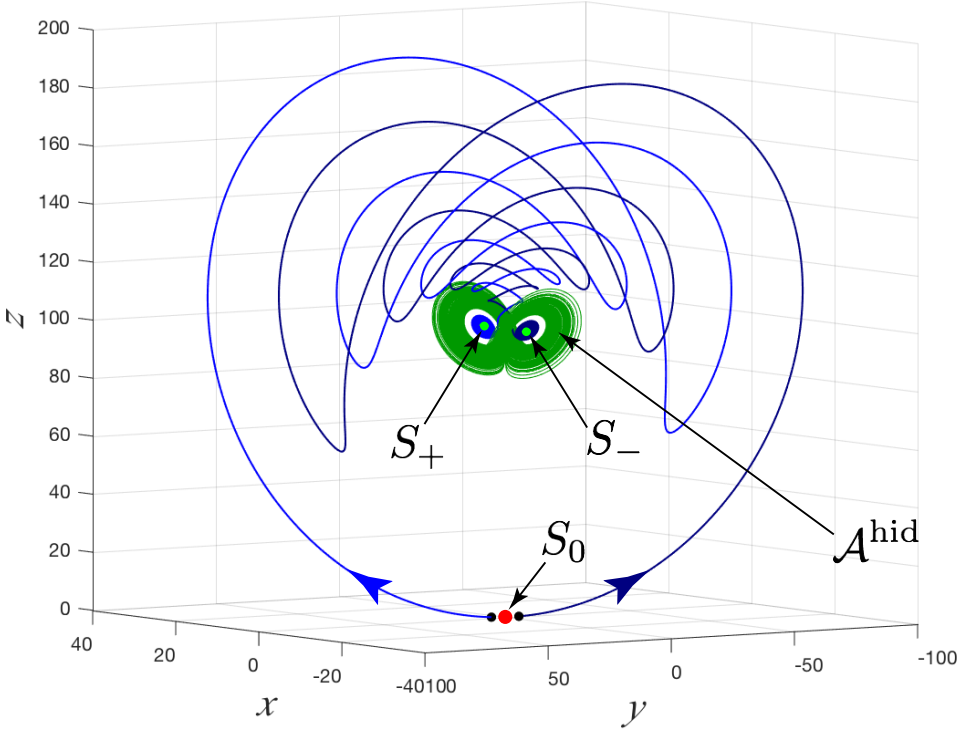
Alexander Gluhovsky, Purdue University, USA

Hidden attractor: Rabinovich model

M.Rabinovich (1978): interactions between waves in plasma ($a < 0$)

$$\dot{x} = -\sigma(x - y) - ayz, \quad \sigma = -ar$$
$$\dot{y} = rx - y - xz$$
$$\dot{z} = -bz + xy$$
$$r = 100, a = -0.009965, b = 0.077454,$$

Equilibria: saddle S_0 & 2 stable $S_\pm$ (light green)
Trajectories from S_0 tend (blue) to zero $S_\pm$;
Hidden attractor (green) coexist with
B-attractor (blue outgoing separatrices of the saddle S_0 attracted to the light green $S_\pm$)



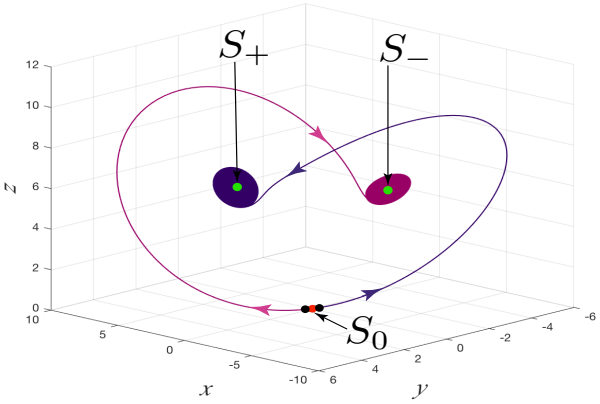
✓ N. Kuznetsov, G. Leonov, T. Mokaev, A. Prasad, M. Shrimali, Finite-time Lyapunov dimension and hidden attractor of the Rabinovich system, **Nonlinear Dynamics**, **2018** http://doi.org/10.1007/s11071-018-4054-z

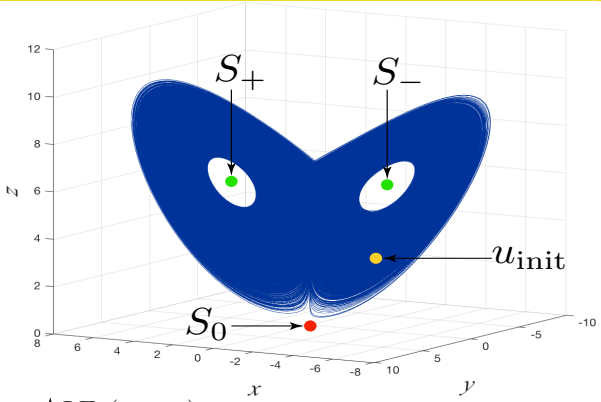
Mikhail I. Rabinovich: BioCircuits Institute, University of California, San Diego, USA

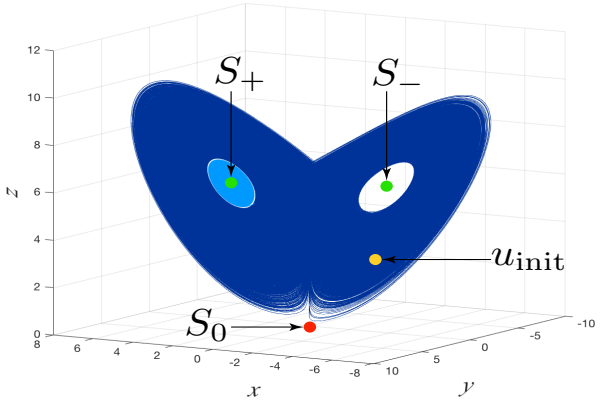
Accuracy of simulation: time of observation and shadowing

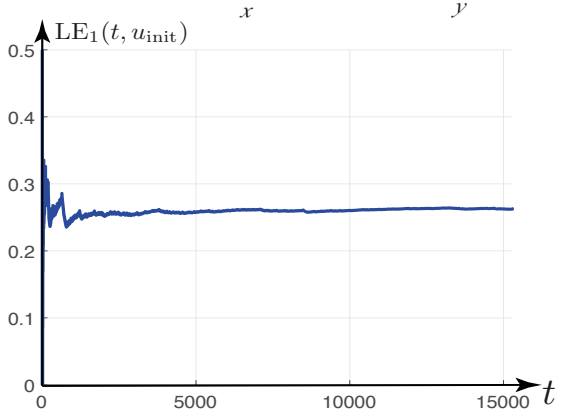
Lorenz-like model of interaction between waves in plasma: **hidden transient chaos**

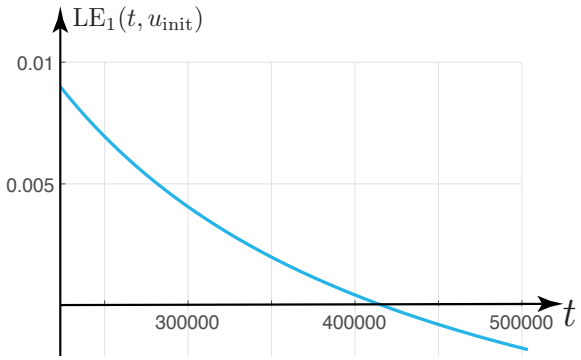
$$\dot{x} = -\sigma(x-y) - ayz,$$
$$\dot{y} = rx - y - xz,$$
$$\dot{z} = -bz + xy, \sigma = -ar,$$
$$r = 6.485, a = -0.5, b = 0.85$$









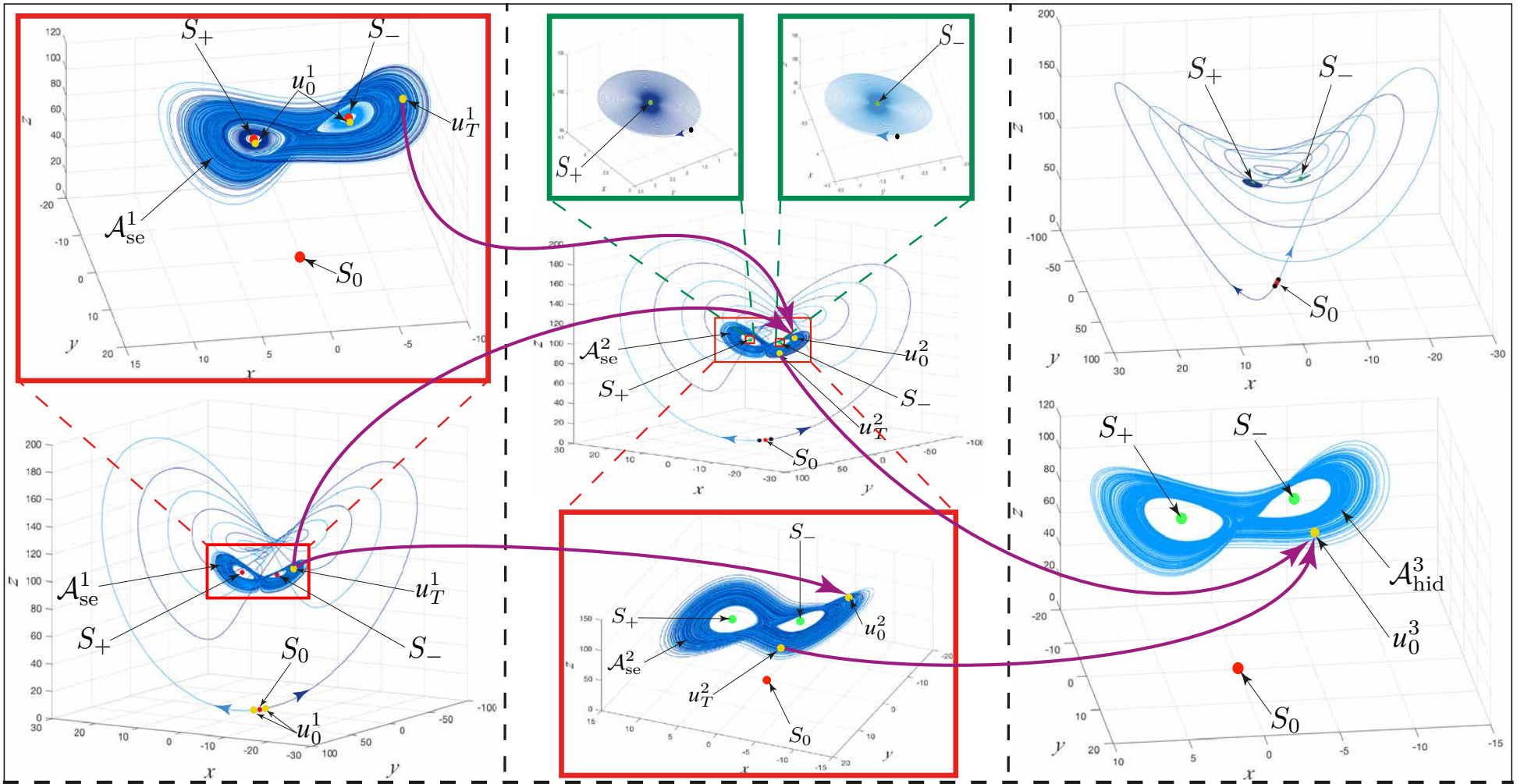


✓ Kehlet, Logg [2015]: for the Lorenz system the time interval of reliable computation (e.g. by ode45) with 16 significant digits and error 10^{-4} is estimated as $[0, 36]$; with error 10^{-8} – $[\mathbf{0, 26}]$!
✓ Liao, Wang [2014]: reliable chaotic solution of Lorenz equation in the interval $[\mathbf{0, 10000}]$ using 3500th-order Taylor expansion and the 4180-digit multiple precision; 1200 CPUs of the National Supercomputer TH-1A; **used CPU time** $\approx$ **221 hours, or 9 days!**

✓ N. Kuznetsov, G. Leonov, T. Mokaev, A. Prasad, M. Shrimali, Finite-time Lyapunov dimension and hidden attractor of the Rabinovich system, **Nonlinear Dynamics**, **2018** http://doi.org/10.1007/s11071-018-4054-z

B. Munmuangsaen, B. Srisuchinwong, A hidden chaotic attractor in the classical Lorenz system, CS&F, **2018**,
Hidden transient chaotic set in the Lorenz systems!, hidden attractor in the Lorenz system is an open question!

Hidden attractors localization by the continuation method



Step 1 : $a = -1.1 \cdot 10^{-2}$, $b = 6.7454 \cdot 10^{-2}$, self-exc. attractor $\mathcal{A}_{se}^1$ w. r. t. equil. $S_0, S_\pm$

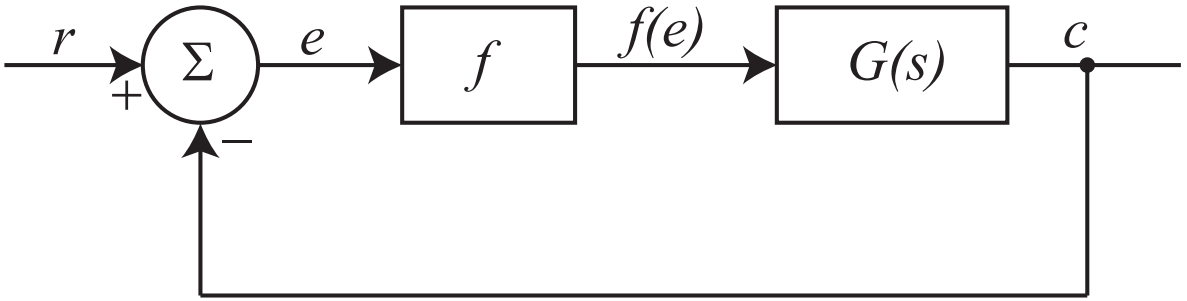
Step 2 : $a = -1.049 \cdot 10^{-2}$, $b = 7.2454 \cdot 10^{-2}$, self-exc. attractor $\mathcal{A}_{se}^2$ w. r. t. equilibrium S_0

Step 3 : $a = -9.965 \cdot 10^{-2}$, $b = 7.7454 \cdot 10^{-2}$, hidden attractor $\mathcal{A}_{hid}^3$, separatrices of $S_0 \to S_\pm$

G. Chen, N.V. Kuznetsov, G.A. Leonov, T.N. Mokaev, Hidden attractors on one path: **Glukhovsky-Dolzhansky**, Lorenz, and **Rabinovich** systems, Int. J. of Bif. and Chaos, 27(8), 2017, art. 1750115

Kalman conjectures, 1957

[1] Kalman R. E., Physical and Mathematical mechanisms of instability in nonlinear automatic control systems, **Transactions of ASME**, 79(3), **1957**, 553-566



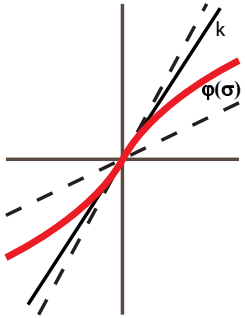


Fig.1. Nonlinear control system. $G(s)$ is a linear transfer function, $f(e)$ is a single-valued, continuous, and differentiable

In 1957 R.E. Kalman wrote [1]: " If $f(e)$ in Fig. 1 is replaced by constants K corresponding to all possible values of $f'(e)$, and it is found that the closed-loop system is stable for all such K, then it is intuitively clear that the system must be monostable; i.e., all transient solutions will converge to a unique, stable critical point."

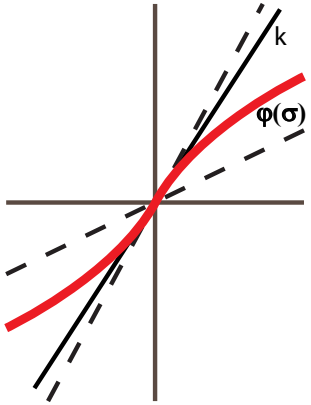
Engineering intuition: the statement is true for models in $\mathbb{R}^{1,2,3}$!

G.A. Leonov, N.V. Kuznetsov, Algorithms for Searching for Hidden Oscillations in the Aizerman and Kalman Problems, **Doklady Mathematics**, 84(1), **2011**, 475-481

In 2013 R.E. Kalman wrote: "*I was far too young and lacking technical mathematical knowledge to go more deeply into the matter.*"

M.Aizerman & R. Kalman problems and harmonic balance

if $\dot{\mathbf{z}}=\mathbf{A}\mathbf{z}+\mathbf{b}k\mathbf{c}^*\mathbf{z}$, is asympt. stable $\forall\, k\in(k_1,k_2):\mathbf{z}(t,\mathbf{z}_0)\to 0\ \forall\mathbf{z}_0$, then is $\dot{\mathbf{x}}=\mathbf{A}\mathbf{x}+\mathbf{b}\varphi(\sigma)$, $\sigma=\mathbf{c}^*\mathbf{x}$, $\varphi(0)=0$, $k_1<\varphi(\sigma)/\sigma<k_2$: $\mathbf{x}(t,\mathbf{x}_0)\to 0\ \forall\mathbf{x}_0$?



$1949:\ k_1<\varphi(\sigma)/\sigma<k_2$

$1957:\ k_1<\varphi'(\sigma)<k_2$

Describing Function Method: the system is monostable (e.g. there are no any periodic solutions) in the case of Aizerman's and Kalman's conditions.

In general, the conjectures are not true: Aizerman's in $\mathbb{R}^2$ and Kalman's in $\mathbb{R}^4$. Counterexamples: the only equilibrium, which is stable, coexist with a stable oscillations (hidden attractor) which basin of attraction is often small.

Survey: Leonov G.A.,Kuznetsov N.V., Hidden attractors in dynamical systems. From hidden oscillations in Hilbert-Kolmogorov, Aizerman, and Kalman problems to hidden chaotic attractor in Chua circuits, Int. J. Bif. and Chaos, 23(1), 2013, art. no. 1330002, doi: 10.1142/S0218127413300024

The history of Kalman's and Aizerman's conjectures

if $\dot{\mathbf{z}}=\mathbf{A}\mathbf{z}+\mathbf{b}k\mathbf{c}^*\mathbf{z}$, is asympt. stable $\forall\, k\in(0,k_2):\mathbf{z}(t,\mathbf{z}_0)\to 0\ \forall\mathbf{z}_0$, then is $\dot{\mathbf{x}}=\mathbf{A}\mathbf{x}+\mathbf{b}\varphi(\sigma)$, $\sigma=\mathbf{c}^*\mathbf{x}, \varphi(0)=0$, $\mathbf{x}(t,\mathbf{x}_0)\to 0\ \forall\mathbf{x}_0$? (trivial global attractor)
M.Aizerman (1949): $0<\varphi(\sigma)/\sigma<k_2 \Leftarrow$ R.Kalman (1957): $0<\varphi'(\sigma)<k_2$

Krasovsky N. 1952 counterexamples to Aizerman's conj. in $\mathbb{R}^2$ ($\mathbf{x}(t)\to+\infty$); V.Pliss

Fitts R. 1966: counterexamples to Kalman's conj. in $\mathbb{R}^4$, nonlinearity $\varphi(\sigma)=\sigma^3$

Barabanov N. 1979-1988: some of Fitts counterexamples are false; analytical 'counterex.' construction in $\mathbb{R}^4$, $\varphi(\sigma)$ close to $\mathrm{sign}(\sigma)$ $(0\le\varphi'(\sigma))$

'gaps' were discussed by Glutsyuk, Meisters, Bernat & Llibre

Yakubovich, Barabanov, Leonov (1965,1988,1996) frequency theorem with $\varphi'(\sigma)\Rightarrow$ Kalman conjecture is true in $\mathbb{R}^3$

Bernat J. & Llibre J. 1996: analytical-numerical 'counterexamples' construction in $\mathbb{R}^4$, $\varphi(\sigma)$ 'close' to $\mathrm{sat}(\sigma)$ $(0\le\varphi'(\sigma))$

Leonov G., Kuznetsov N. 2011: some of Fitts counterexamples are true; smooth counterex. in $\mathbb{R}^4$: $\varphi(\sigma)=\tanh(\sigma)$: $0<\tanh'(\sigma)\leqslant 1<k_2=9.9$; effective analytical-numerical counterexamples construction

Carrasco J. et al. 2014: Discrete-time Kalman conjecture is false in $\mathbb{R}^2$

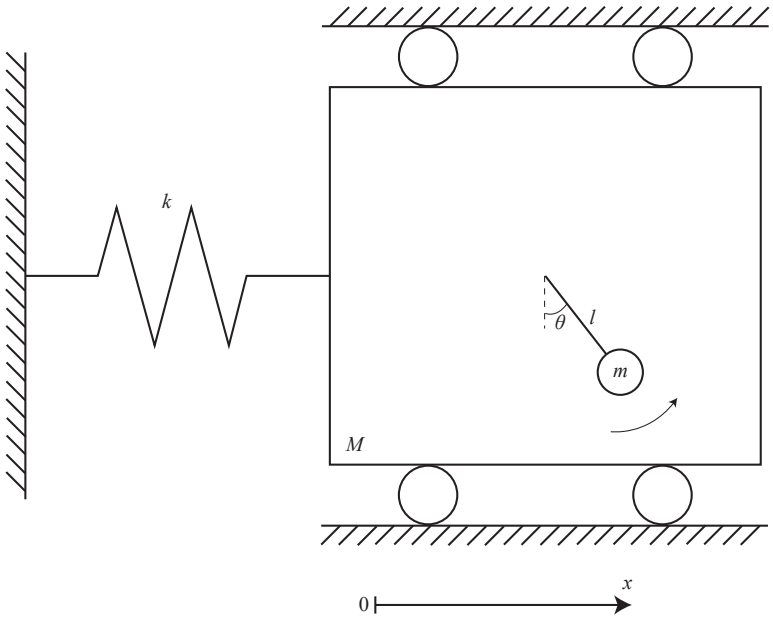
Survey: V.O. Bragin, V.I. Vagaitsev, N.V. Kuznetsov, G.A. Leonov (2011) Algorithms for finding hidden oscillations in nonlinear systems. The Aizerman and Kalman conjectures and Chua's circuits, *J. of Computer and Systems Sciences Int.*, 50(4), 511-544 (doi:10.1134/S106423071104006X)

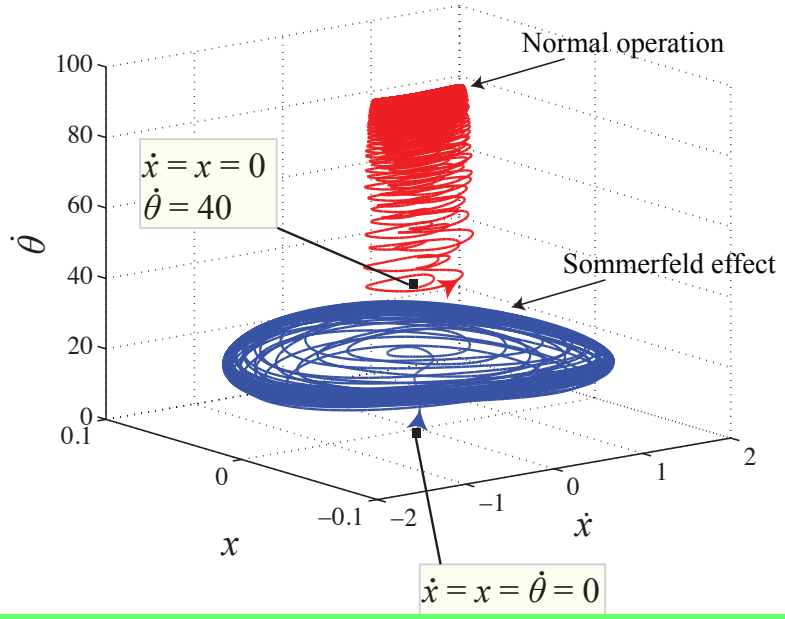
Hidden attractors in engineering problems: systems without equilibria

A. Sommerfeld effect (1902): represents the inability of a system to be spun up by a torque-limited rotor to a desired rotational velocity due to its resonant interaction with another part of the system. [e.g. engine start on submarine in the water]

Support (cart) is attached to wall by spring. Rotating eccentric mass (unbalanced rotor) is connected to the cart is actuated by DC motor.

$$(M+m)\ddot{x}+k_1\dot{x}+ml(\ddot{\theta}\cos\theta-\dot{\theta}^2\sin\theta)+kx=0, \qquad J\ddot{\theta}+k_\theta\dot{\theta}+ml\ddot{x}\cos\theta=u$$





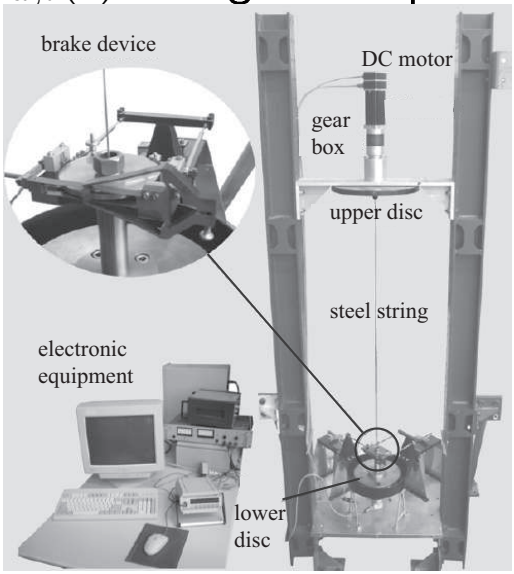
Kiseleva M.A., Kuznetsov N.V., Leonov G.A., Hidden attractors in electromechanical systems with and without equilibria, **IFAC-PapersOnLine**, 49(14), **2016**, 51-55

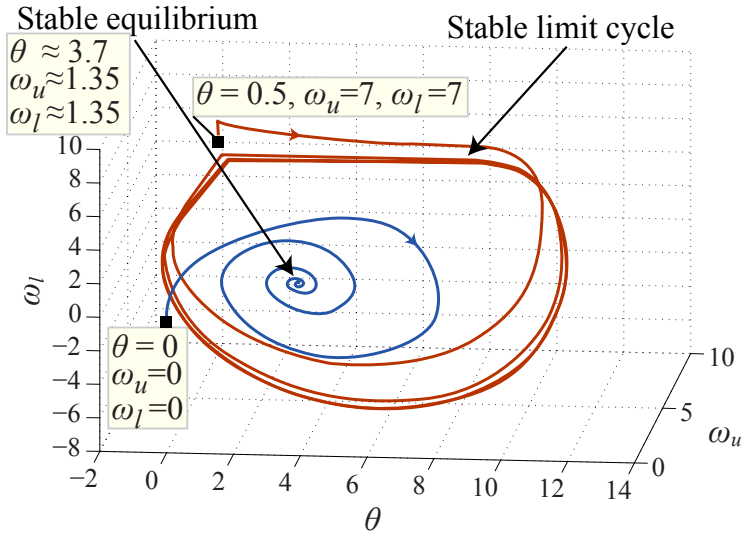
Hidden attractors in engineering problems

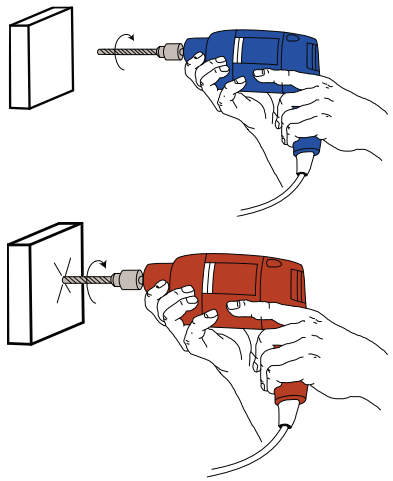
Drilling setup in Eindhoven TU: de Bruin et al. 2004

$$J_u\ddot{\theta}_u+k_\theta(\theta_u-\theta_l)+b(\dot{\theta}_u-\dot{\theta}_l)+T_{fu}(\dot{\theta}_u)-k_m v=0,\ \ J_l\ddot{\theta}_l-k_\theta(\theta_u-\theta_l)-b(\dot{\theta}_u-\dot{\theta}_l)+T_{fl}(\dot{\theta}_l)=0$$

$\theta_{u,l}(t)$ – angular displacements of the upper & lower discs, $T_{fu,l}$ – friction







YF22 Raptor Boeing crash in 1992: Lauvdal et al. 1997 CDC

"Since stability in simulations does not imply stability of the physical control system (an example is the crash of the YF22) stronger theoretical understanding is required"

Aircraft control system — Kalman conjecture

Hidden oscillations in 16th Hilbert problem

1900: 16th Hilbert problem (second part)

What are the number and mutual disposition of limit cycles for

$$\dot{x} = P_n(x,y) = a_1x^2 + b_1xy + c_1y^2 + \alpha_1x + \beta_1y + \ldots$$
$$\dot{y} = Q_n(x,y) = a_2x^2 + b_2xy + c_2y^2 + \alpha_2x + \beta_2y + \ldots$$

N.N. Bautin 1949-1952: 3 limit cycles (LCs) [around one focus]

I.G. Petrovskii, E.M. Landis 1955–1959: only 3 LCs

L. Chen & M. Wang, S. Shi 1979-80: 4 LCs [(1,3), 2 focuses]

Y. Ilyashenko 1995: finiteness of the number of limit cycles

P. Zhang 2001: two focuses $\Rightarrow$ only (1,n) distribution of nested cycles

number of limit cycles $H(n)$: $H(2) \geqslant 4$

What about visualization of limit cycles?

16th Hilbert problem: Poincare visualization problem

Poincare visualization problem, 1881: *"the problem ... to construct the curves defined by differential equations..."*

16th Hilbert problem (second part) 1900:
number and mutual disposition of limit cycles for

$$\dot{x} = P_n(x,y) = a_1x^2 + b_1xy + c_1y^2 + \alpha_1x + \beta_1y + ...$$
$$\dot{y} = Q_n(x,y) = a_2x^2 + b_2xy + c_2y^2 + \alpha_2x + \beta_2y + ...$$

V. Arnold (2005): To estimate the number of LCs of square vector fields on plane, academician A.N. Kolmogorov had distributed several hundreds of such fields (with randomly chosen coefficients of quadratic expressions) among a few hundreds of students of Mech.&Math. Faculty of Moscow Univ. as a mathematical practice. Each student had to find the number of LCs of his/her field. The result of this experiment was absolutely unexpected: not a single field had a LC!...

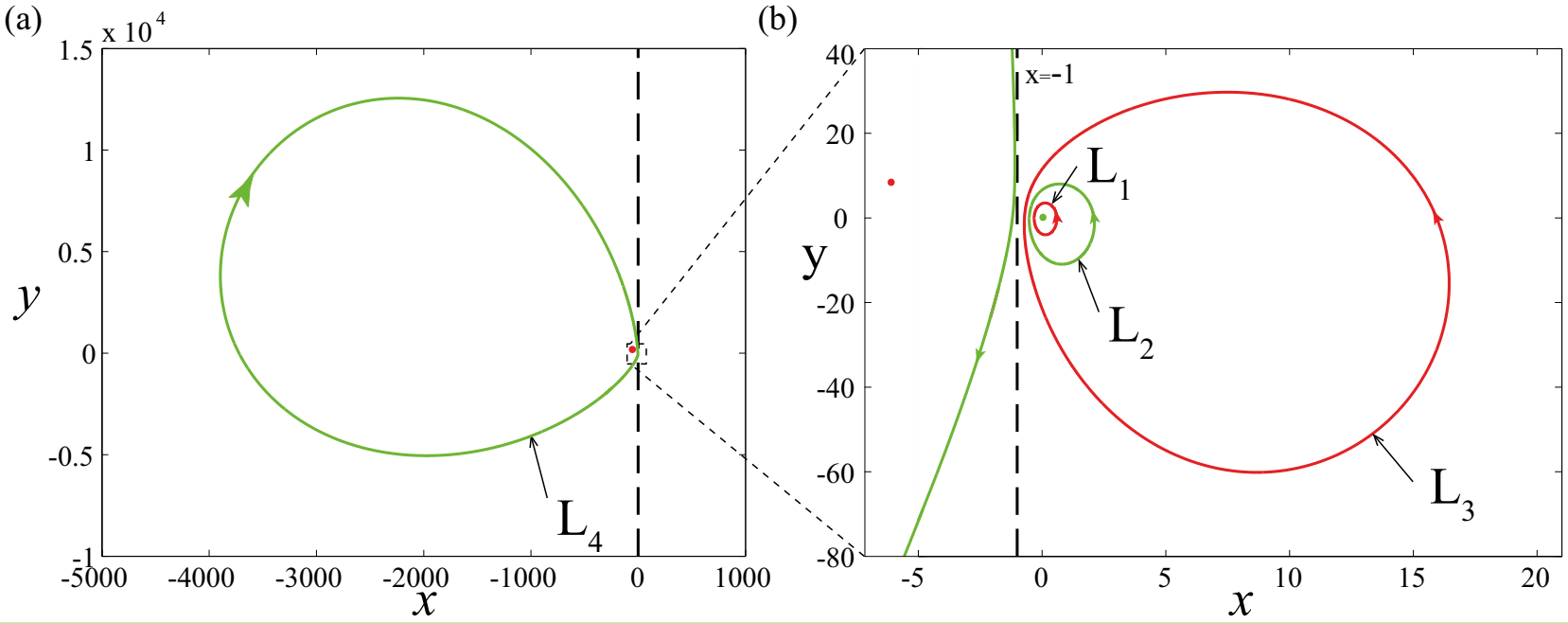
Visualization problem: nested limit cycles are hidden oscillations

Hidden oscillations in 16th Hilbert problem

D.Hilbert (1900): number & mutual disposition of limit cycles

$$\dot{x} = P_n(x,y) = a_1x^2 + b_1xy + c_1y^2 + \alpha_1x + \beta_1y + \ldots$$
$$\dot{y} = Q_n(x,y) = a_2x^2 + b_2xy + c_2y^2 + \alpha_2x + \beta_2y + \ldots$$

In the right subfig there are 3 nested limit cycles around stable zero equilibrium.
Red – unstable limit cycles, green – stable limit cycles and equilibrium:
stable equilibrium coexists with stable limit cycle – a hidden oscillation L_2.



N.V. Kuznetsov, O.A. Kuznetsova, G.A. Leonov, Visualization of four normal size limit cycles in two-dimensional polynomial quadratic system, Differential equations and Dynamical systems, 21(1-2), 2013, 29-34 (doi:10.1007/s12591-012-0118-6)

Hidden attractors and multistability

✓ A. Jenkins, Self-oscillation, **Physics Reports**, 525(2), 2013
✓ A.N. Pisarchik, U. Feudel, Control of multistability, **Physics Reports**, 540(4), 2014
✓ D. Dudkowski, S. Jafari, T. Kapitaniak, N. V. Kuznetsov, G. A. Leonov, A. Prasad, Hidden attractors in dynamical systems, **Physics Reports**, 637, 2016
✓ X. Zhang, G. Chen, Constructing an autonomous system with infinitely many chaotic attractors, **Chaos**, 27(7), 2017 art. num. 071101
✓ Brzeski P., Wojewoda J., Kapitaniak T., Kurths J., and Perlikowski P., Sample-based approach can outperform the classical dynamical analysis - experimental confirmation of the basin stability method, **Scientific Reports**, 7, 2017, art. num. 6121
✓ Garashchuk I., Sinelshchikov D., Kudryashov N., Hidden attractors in a model of a bubble contrast agent oscillating near an elastic wall, **EPJ Web of Conferences**, 173, 2018, art. num. 06006
✓ Kuznetsov A., Kuznetsov S., Mosekilde E., Stankevich N., Co-existing hidden attractors in a radio-physical oscillator system, **J. of Physics A: Mathematical and Theoretical**, 48, 2015, art. num. 125101
✓ Zhusubaliyev Z., Mosekilde E., Churilov A., Medvedev A., Multistability and hidden attractors in an impulsive Goodwin oscillator with time delay, **Eur. Phys. J. Special Topics**, 224(8), 2015, 1519-1539
✓ Burkin I., Hidden attractors of some multistable systems with infinite number of equilibria, **Chebyshevskii Sb.**, 18(2), 2017, 18-33
✓ Wei Z., Moroz I., Sprott J., Akgul A., Zhang W., Hidden hyperchaos and electronic circuit application in a 5D self-exciting homopolar disc dynamo, **Chaos**, 27(3), 2017, art. num. 033101
✓ Semenov V., Korneev I., Arinushkin P., Strelkova G., Vadivasova T., Anishchenko V., Numerical and experimental studies of attractors in memristor-based Chua's oscillator with a line of equilibria. Noise-induced effects, **European Physical Journal: Special Topics**, 224(8), **2015**, 1553-1561,

Hidden attractors 2009-till now in: Glukhovsky-Dolzhansky system (convective fluid motion inside a rotating ellipsoidal cavity), Rabinovich system (interaction between waves in plasma), Nose-Hoover system (oscillator), Rabinovich-Fabricant system (the Tollmien-Schlichting waves in hydrodynamic flows), Sprott system, Chua electronic circuits, memristors, rotating electromechanical systems with Sommerfeld effect, phase-locked loops, drilling systems, aircraft control systems...

Hidden attractors

✓ Leonov G.A.,Kuznetsov N.V., Hidden attractors in dynamical systems. From hidden oscillations in Hilbert-Kolmogorov, Aizerman, and Kalman problems to hidden chaotic attractor in Chua circuits, **Int. J. of Bif. and Chaos**, 23(1), 2013, art. no. 1330002
✓ Leonov G.A., Kuznetsov N.V., Mokaev T.N., Homoclinic orbits, and self-excited and hidden attractors in a Lorenz-like system describing convective fluid motion, **European Phys. J. Special Topics**, 224, 2015, 1421-1458
✓ N.V. Stankevich, N.V. Kuznetsov, G.A. Leonov, L. Chua, Scenario of the birth of hidden attractors in the Chua circuit, **Int. J. of Bif. and Chaos**, 27(12), 2017, art. num. 1730038
✓ G. Chen, N. Kuznetsov, G. Leonov, T. Mokaev, Hidden attractors on one path: Glukhovsky-Dolzhansky, Lorenz, and Rabinovich systems, **Int. J. of Bif. and Chaos**, 27(8), 2017 art. num. 1750115
✓ M.-F. Danca, N.V. Kuznetsov, Hidden chaotic sets in a Hopfield neural system, **Chaos, Solitons, and Fractals**, vol. 103, 2017, 144-150
✓ Kuznetsov N.V., Leonov G.A., Yuldashev M.V., Yuldashev R.V., Hidden attractors in dynamical models of phase-locked loop circuits: limitations of simulation in MATLAB and SPICE, **Commun Nonlinear Sci Numer Simulat**, vol. 51, 2017, 39-49
✓ M.-F. Danca, N. Kuznetsov, G. Chen, Unusual dynamics and hidden attractors of the Rabinovich-Fabrikant system, **Nonlinear Dynamics**, 88(1), 2017, 791-805
✓ Kuznetsov N.V., Hidden attractors in fundamental problems and engineering models. A short survey, **Lecture Notes in Electrical Engineering**, vol. 371, 2016, 13-25
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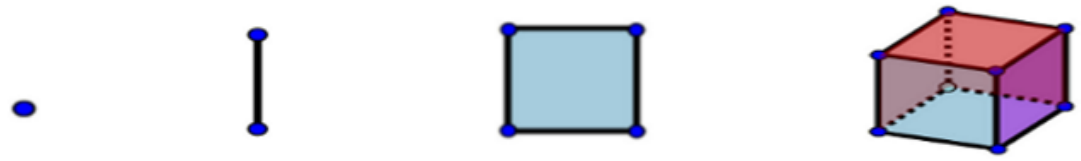
✓ Sharma P.R., Shrimali M.D., Prasad A., Kuznetsov N.V., Leonov G.A., Controlling dynamics of hidden attractors, **Int. J. of Bif. and Chaos**, 25, 2015, art. num. 1550061
✓ Leonov G.A., Kuznetsov N.V., Kiseleva M.A. et al., Hidden oscillations in mathematical model of drilling system actuated by induction motor with a wound rotor, **Nonlinear Dynamics**, 77(1-2), 2014, 277-288
✓ N.V. Kuznetsov, G.A. Leonov, Hidden attractors in dynamical systems: systems with no equilibria, multistability and coexisting attractors, **IFAC Proceedings Volumes**, 47(3), 2014, 5445-5454
✓ V.O. Bragin, V.I. Vagaitsev, N.V. Kuznetsov, G.A. Leonov, Algorithms for Finding Hidden Oscillations in Nonlinear Systems. The Aizerman and Kalman Conjectures and Chua's Circuits, **J. of Computer and Systems Sciences Int.**, 50(4), 2011, 511-543

Content

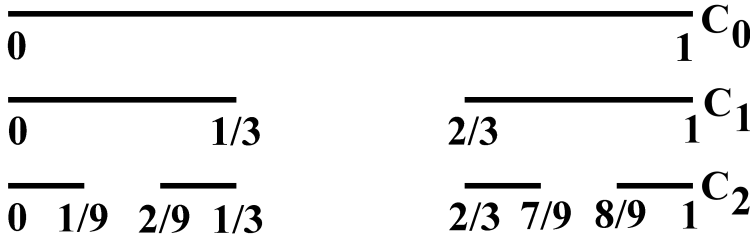
Dimensions of sets

Topological dimension of K ($\dim_{\mathrm{T}} K$) is the smallest **integer** n such that each point $p \in K$ has a small neighborhood, the boundary of which has dimension $< n$ (H.Poincare)



$\dim_{\mathrm{T}} P = 0$ $\dim_{\mathrm{T}} L = 1$ $\dim_{\mathrm{T}} S = 2$ $\dim_{\mathrm{T}} C = 3$

Cantor dust $\mathcal{C} := \cap_{k\in\mathbb{N}} C_k$



$\dim_{\mathrm{T}} \mathcal{C} = 0, \quad \dim_{\mathrm{H}} \mathcal{C} = \ln 2/\ln 3$

Consider a covering of K by balls with diameter $r_i < \varepsilon$. Then **Hausdorff dimension** ($\dim_{\mathrm{H}} K$) is number d, such that $\lim_{\varepsilon\to 0} \inf \sum_i r_i^d \neq 0, \neq \infty$; can take any **nonnegative value**; suits for study of **fractal sets**.

$\mathcal{N}_\varepsilon(K)$ the minimal number of balls of radius ε needed to cover a bounded set $K \subset \mathbb{R}^n$. Consider the numbers $d \geq 0$, $\varepsilon > 0$, and put $\mu_F(K,d,\varepsilon) = \mathcal{N}_\varepsilon(K)\varepsilon^d$, $\mu_F(K,d) = \limsup_{\varepsilon\to 0} \mu_F(K,d,\varepsilon)$.

Fractal dimension: $\dim_{\mathrm{F}} K = \inf\{d \geq 0 \mid \mu_F(K,d) = 0\}$

$\dim_{\mathrm{T}} K \leq \dim_{\mathrm{H}} K \leq \dim_{\mathrm{F}} K \leq$ Lyapunov dimension

Attractors with noninteger Hausdorff dimension are called strange attractors

Hausdorff-Lebesgue dimension of sets

✓ Leonov G.A., Hausdorff-Lebesgue dimension of attractors, **Int. Journal of Bifurcation and Chaos**, 27(10), **2017**, art. num. 1750164

Consider all coverings of compact K by disjoint cubes C_i with sides $2\delta_i \le 2\varepsilon$.
$\mu_{HL}(K,d,\varepsilon) = \inf \sum\limits_{I} \delta_i^d$, where the infimum is taken over all 2ε-coverings of K.
$\mu_{HL}(K,d,\varepsilon)$ increases with decreasing $\varepsilon \Rightarrow \mu_{HL}(K,d) = \lim\limits_{\varepsilon\to 0} \mu_{HL}(K,d,\varepsilon)$.
Def. The value $\mu_{HL}(K,d)$ is called a Hausdorff–Lebesgue measure of compact K.
We introduce $\dim_{\mathrm{HL}} K = \inf\{d \,|\, \mu_{\mathrm{HL}}(K,d) = 0\}$.
Def. The value $\dim_{\mathrm{HL}} K$ is called a Hausdorff–Lebesgue dimension of compact K.
Consider now all coverings of K by disjoint cubes C_i with sides 2ε
Def. $\mu_{\mathrm{FHL}}(K,d) = \limsup\limits_{\varepsilon\to 0} \sum\limits_{i} \varepsilon^d$ is called a Hausdorff–Lebesgue fractal measure like functional of compact K.
Def. Hausdorff–Lebesgue fractal dimension $\dim_{\mathrm{FHL}} K = \inf\{d \,|\, \mu_{\mathrm{FHL}}(K,d) = 0\}$
$\mu_{\mathrm{H}}(K,d,\sqrt{n}\varepsilon) \le \mu_{\mathrm{HL}}(K,d,\varepsilon)(n)^{\frac{d}{2}}$,
$\mu_{\mathrm{H}}(K,d) \le \mu_{\mathrm{HL}}(K,d)(n)^{d/2} \le \mu_{\mathrm{FHL}}(K,d)(n)^{\frac{d}{2}}$,

$\dim_{\mathrm{HK}} K \le \dim_{\mathrm{HL}} K \le \dim_{\mathrm{FHL}} K \le$ Lyapunov dimension

Finite-time Lyapunov dimension (FTLD)

✓ N. Kuznetsov, G. Leonov, T. Mokaev, A. Prasad, M. Shrimali, **Finite-time Lyapunov dimension** and hidden attractor of the Rabinovich system, **Nonlinear Dynamics**, **2018** http://doi.org/10.1007/s11071-018-4054-z
✓ Kuznetsov N.V., The Lyapunov dimension and its estimation via the Leonov method, **Physics Letters A**, 380(25-26), **2016**, 2142-2149

$(\{\varphi^t\}_{t\geq 0},(U\subseteq\mathbb{R}^n,||\cdot||))$, $D\varphi^t(u)$ singular val: $\sigma_1(t,u)\geq..\geq\sigma_n(t,u)>0$
$\omega_d(D\varphi^t(u))) = \sigma_1(t,u)\cdot\cdot\sigma_{\lfloor d\rfloor}(t,u)\sigma_{\lfloor d\rfloor+1}(t,u)^{d-\lfloor d\rfloor}$, $d\in[0,n)$

Def. Finite-time local Lyapunov Dimension (i.e. LD of φ^t at $u\in U$):
$\dim_{\mathrm{L}}(\varphi^t,u)=\sup\{d\in[0,n]:\omega_d(D\varphi^t(u))\geq 1\}$.

Def. Finite-time Lyapunov Dimension (LD of map φ^t) with respect to $K\equiv\varphi^t(K)$: $\dim_{\mathrm{L}}(\varphi^t,K)=\sup_{u\in K}\dim_{\mathrm{L}}(\varphi^t,u)$

Douady-Oesterlé,1980: $\dim_{\mathrm{H}} K\leq\dim_{\mathrm{L}}(\varphi^t,K)$, $\forall t\geq 0$, K-compact

finite-time Lyapunov exponents: $\mathrm{LE}_i(t,u)=\frac{1}{t}\ln\sigma_i(t,u)$
analog of Kaplan-Yorke formula for the ordered by decreasing set $\{\mathrm{LE}_i(t,u)\}_1^n$:
$d_{\mathrm{L}}^{\mathrm{KY}}(\mathrm{LE}_i(t,u)\}_1^n)=j(t,\mathrm{u})+\frac{\mathrm{LE}_1(t,u)+\cdots+\mathrm{LE}_{j(t,u)}(t,u)}{|\mathrm{LE}_{j(t,u)+1}(t,u)|}$,
$j(t,u)=\max\{m:\sum_1^m\mathrm{LE}_i(t,\mathrm{x})\geq 0\},\sum_1^{m+1}\mathrm{LE}_i(t,u)<0$

$\dim_{\mathrm{L}}(\varphi^t,K)=\sup_{u\in K}\dim_{\mathrm{L}}(\varphi^t,u)=\sup_{u\in K}d_{\mathrm{L}}^{\mathrm{KY}}(\mathrm{LE}_i(t,u)\}_1^n)$

Lyapunov dimension (via finite-time LD) and its invariance

Lemma. $\inf_{t\geq 0}\dim_{\mathrm{L}}(\varphi^t,u)=\liminf_{t\to+\infty}\sup_{u\in K}\dim_{\mathrm{L}}(\varphi^t,u)$.

Def. Lyapunov dimension of dynamical system $\{\varphi^t\}_{t\geq 0}$ with respect to K:
$\dim_{\mathrm{L}}(\{\varphi^t\}_{t\geq 0},K)=\liminf_{t\to+\infty}\sup_{u\in K}\dim_{\mathrm{L}}(\varphi^t,u)$

Lemma. $\dim_{\mathrm{L}}(\{\varphi^t\}_{t\geq 0},K)=\liminf_{t\to+\infty}\sup_{u\in K}d_{\mathrm{L}}^{\mathrm{KY}}(\{\mathrm{LE}_i(t,u)\}_1^n)$

Hausdorff dimension is invariant with respect to Lipschitz diffeomorphisms.

Diffeomorphism $h\colon U\subseteq\mathbb{R}^n\to\mathbb{R}^n$, smooth change of coordinates $w=h(u)$
Dynamical system: $\left(\{\varphi^t\}_{t\geq 0},(U\subseteq\mathbb{R}^n,||\cdot||)\right)\to\left(\{\varphi_h^t\}_{t\geq 0},(h(U)\subseteq\mathbb{R}^n,||\cdot||)\right)$
$K\equiv\varphi^t(K)\subset U\to h(K)\equiv\varphi_h^t(h(K))\subset h(U)$ $D\varphi_h^t(w)=Dh(\varphi^t(u))D\varphi^t(u)(Dh(u))^{-1}$

Lemma. Lyapunov dimension of dynamical system $\dim_{\mathrm{L}}(\{\varphi^t\}_{t\geq 0},K)$ is invariant under diffeomorphism: $\dim_{\mathrm{L}}(\{\varphi^t\}_{t\geq 0},K)=\dim_{\mathrm{L}}(\{\varphi_h^t\}_{t\geq 0},h(K))$;
$\limsup_{t\to+\infty}\mathrm{LE}_i\left(D\varphi_h^t(h(u))\right)=\limsup_{t\to+\infty}\mathrm{LE}_i\left(D\varphi^t(u)\right),i=1,2,..,n.$

✓ Kuznetsov N.V., Alexeeva T.A., Leonov G.A., Invariance of Lyapunov exponents and Lyapunov dimension for regular and irregular linearizations, **Nonlinear Dynamics**, 85(1), 2016, 195-201
✓ Kuznetsov N.V., The Lyapunov dimension and its estimation via the Leonov method, **Physics Letters A**, 380(25-26), **2016**, 2142-2149

Lyapunov dimension: analytical estimation

G. Leonov, **1991**: On estimations of Hausdorff dimension of attractors, Vestnik St. Petersburg University: Mathematics, 24(3)

✓ Leonov G.A., Boichenko V.A., Lyapunov's direct method in the estimation of the Hausdorff dimension of attractors, **Acta Applicandae Mathematicae**, 26(1), **1992** 1-60.
✓ Kuznetsov N.V., The Lyapunov dimension and its estimation via the Leonov method, **Physics Letters A**, 380(25-26), **2016**, 2142-2149

$Dh(u)=e^{V(u)(j+s)^{-1}}S$: $V:U\subseteq\mathbb{R}^n\to\mathbb{R}^1$ smooth, nonsingular matrix S
$\lambda_1(u,S)\geq\cdot\cdot\geq\lambda_n(u,S)$ eigenvalues $\frac{1}{2}(SJ(u(t,u))S^{-1}+(SJ(u(t,u))S^{-1})^*)$

Theorem. (1991) If for integer $j\in[1,n]$ and real $s\in[0,1)$
$\exists$ smooth $V:\mathbb{R}^n\to\mathbb{R}$ and nonsingular matrix S:
$\lambda_1(u,S)+..+\lambda_j(u,S)+s\lambda_{j+1}(u,S)+\dot{V}(u)<0$, $\forall u\in K$ then $\dim_{\mathrm{L}}(\{\varphi^t\}_{t\geq 0},K)\leqslant j+s$

discrete-time: $\lambda_1(u,S)\geq.\geq\lambda_n(u,S)$ eigenval. of $(SJ(u(t,u))S^{-1})^*SJ(u(t,u))S^{-1}$

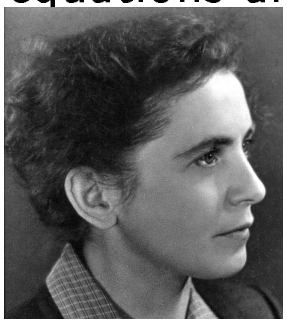
Theorem. (2016) $\ln\lambda_1(u,S)+..+\ln\lambda_j(u,S)+s\ln\lambda_{j+1}(u,S)+\big(V(\varphi(u))-V(u)\big)$, $\forall u\in K$ then $\dim_{\mathrm{L}}(\{\varphi^t\}_{t\geq 0},K)\leqslant j+s$

If the estimation is valid $\forall u\in U\Rightarrow$ the localization of attractor in not needed!

Thm. If at an equilibrium u_{eq} the Jacobian $D\varphi(u_{eq})$ has simple real eigenvalues: $\{\lambda_i(u_{eq})\}_{i=1}^n$, $\lambda_i(u_{eq})\geq\lambda_{i+1}(u_{eq})$, then $\dim_{\mathrm{L}}u_{eq}=d_{\mathrm{L}}^{\mathrm{KY}}(\{\lambda_i(u_{eq})\}_{i=1}^n)$

Exact Lyapunov dimension formulas for the global attractors

A.B. Babin, M.I. Vishik, **1983** Attractors of evolutionary partial differential equations and estimates of their dimension. Usp Mat Nauk, 38(4), 133-87



O.A. Ladyzhenskaya, **1987**. Finding the minimal global attractors for the Navier-Stokes equations and other equations with partial derivatives, Usp Mat Nauk, 42(6), 25-60: $\dim_{\mathrm{H}} \mathbf{K} < \infty$
global attractor: an invariant, closed, uniformly attracting set in the phase space of the dynamical system

Landau-Hopf conjecture 1944, 1948: turbulence developed as a series of bifurcations of quasi-periodic solution with increasing frequency $T, T^2 \ldots$ (soft excitation)
Lorenz 1963 chaos: truncated Galerkin approx. of Rayleigh-Benard fluid convection
Ruelle-Takens 1971: Series may stop at T^4 & then a strange attractor appear (hard excitation – i.e. finite dimensional dynamics may explain chaos & turbulence).

C.R. Doering et al., **1986**: **Exact Lyapunov Dimension** of the universal attractor for the complex Ginzburg-Landau equation, Phys.Rev.Lett. 59.
A. Eden conjecture, **1989**: the maximum of the local LD is achieved on an equilibrium (e.g. LD the global Lorenz attractor achieved at 0 equilibrium)

Conjecture, 2016: the Lyapunov dimension of **(local) self-excited attractor** is **less than the Lyapunov dimension of one of the unstable equilibria**, the unstable manifold of which intersects with the basin of attraction and visualize the attractor.

Exact Lyapunov dimension of one Chua memristor circuit.

F. Corinto, M. Forti, Memristor circuits: bifurcations without parameters, IEEE Transactions on Circuits and Systems I: Regular Papers 64 (6) (2017) 1540-1551.

$\dot{x}=\alpha(m_0-1)x+\alpha y-\alpha m_1 x^3+\alpha x_0, \dot{y}=x-y+z,\ \dot{x}=\beta y-\gamma z$

Linearization $\dot{v} = J(\varphi^t(u))v, \quad J(u) = Df(u),$

$$J(u) = \begin{pmatrix} \alpha(m_0-1)-3\alpha m_1 x^2 & \alpha & 0 \\ 1 & -1 & 1 \\ 0 & \beta & -\gamma \end{pmatrix} = J(0) - 3\alpha m_1 x^2 I_1$$

$$J_0 = J(0) = \begin{pmatrix} \alpha(m_0-1) & \alpha & 0 \\ 1 & -1 & 1 \\ 0 & \beta & -\gamma \end{pmatrix},\ I_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Let for any $t > 0$ and any $u \in U$ the ordered sequence $\lambda_1(u) \geq \cdots \geq \lambda_n(u)$, where $\lambda_i(u) = \lambda_i(\frac{1}{2}(J(u)+J(u)^*)$, $i = 1, \ldots, n$ be the eigenvalues of the symmetrized Jacobian matrix $\frac{1}{2}\left(J(u)+J(u)^*\right)$. Lemma: $\lambda_j(0) \geq \lambda_j(u), \quad j = 1, 2, 3$

Theorem: $\dim_{\mathrm{L}} K = d_{\mathrm{L}}^{\mathrm{KY}}\left(\{\lambda_j(u)\}_{i=1}^3\right) \leq d_{\mathrm{L}}^{\mathrm{KY}}\left(\{\lambda_j(0)\}_{i=1}^3\right) = \dim_{\mathrm{L}}(K \ni 0)$;
If $\lambda_1(0)+\lambda_2(0) < 0 \Rightarrow$ convergency to the stationary set

G. Leonov, N. Kuznetsov, The Lyapunov dimension, convergency and entropy for a dynamical model of Chua memristor circuit, 2018, https://arxiv.org/pdf/1801.09679.pdf

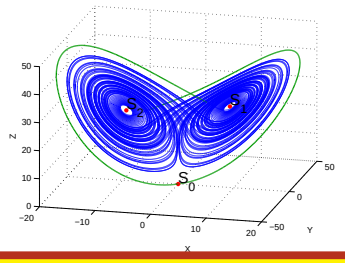
Exact Lyapunov dimension of the Lorenz attractor:
complete solution of the Eden problem on the Lorenz system

A. Eden, C. Foias, R. Temam (1991), $2.401 < \dim_{\mathrm{L}} K < 2.409$

G. Leonov, 1991: On estimations of Hausdorff dimension of attractors, Vestnik St. Petersburg University: Mathematics, 24(3)
allows to estimate without localization of the attractor in the phase space

Lorenz system: $r, \sigma, b > 0$ – parameters

$$\begin{aligned} \dot{x} &= -\sigma x + \sigma y, \\ \dot{y} &= rx - y - xz, \\ \dot{z} &= -bz + xy, \end{aligned}$$



Thm (Leonov, 2002, 2017). If $\frac{2(\sigma+b+1)}{\sigma+1+\sqrt{(\sigma-1)^2+4r\sigma}} \leq 1$, then

$$\dim_{\mathrm{L}} K = 3 - \frac{2(\sigma+b+1)}{\sigma+1+\sqrt{(\sigma-1)^2+4r\sigma}} = 2.401312763583084...,$$

otherwise any solution of the Lorenz system tends to an equilibrium as $t \to +\infty$.

2002-till now: Henon map, Chirikov map, Lozi map, Lorenz system, Tigan system, Shimizu-Morioka system, Glukhovsky-Dolzhansky system, and others

Lyapunov exponents: finite-time numerical computation

P. Cvitanović [Georgia Tech] **Chaos: Classical and Quantum (http://ChaosBook.org)**:
"Whatever you call your exponents, please state clearly how are they being computed".

ODE + variational eq. $\Rightarrow$ **fundamental matrix**:

$$\begin{cases} \dot{u}(s,u_0) = f(u(s,u_0)), \ \ u(0,u_0) = u_0 \in U, \\ \dot{\Phi}(s,u_0) = J(u(s,u_0))\ \Phi(s,u_0), \ \ \Phi(0,u_0) = I. \end{cases}$$

Finite-time LEs: $\mathrm{LE}_i(t,u_0) = \frac{1}{t}\ln\sigma_i(t,u_0)$

$$\Phi(t,u_0) \overset{\text{SVD}}{=} U(t,u_0)\Sigma(t,u_0)\mathrm{V}^*(t,u_0),$$

- $\Sigma(t,u_0) = \mathrm{diag}\{\sigma_1(t,u_0),...,\sigma_n(t,u_0)\}$,
- $U(t,u_0)^*U(t,u_0) \equiv I \equiv \mathrm{V}(t,u_0)^*\mathrm{V}(t,u_0)$,

$$\Phi(t,u_0) \overset{\text{QR}}{=} Q(t,u_0)R(t,u_0),$$

- $R(t,u_0)$ is upper-triangular, $R[i,i] > 0$,
- $Q(t,u_0)^*Q(t,u_0) \equiv I$

To avoid exponential growth of the values: $(0,T] = (0,\tau] \cup (\tau,2\tau] \cup \cdots \cup ((k-1)\tau, k\tau = T]$.

$$\Phi(k\tau,u_0) = \Phi(\tau,u_{k-1})..\Phi(\tau,u_1)\boxed{\Phi(\tau,u_0)} = \Phi(\tau,u_{k-1})..\boxed{\Phi(\tau,u_1)Q_1^0}R_1^0 = .. \overset{\text{QR}}{=} \underbrace{Q_k^0}_{Q}\underbrace{R_k^0..R_1^0}_{R}$$

SVD approximation

$$\Sigma^0 = \Phi(k\tau,u_0)^*\,Q_k^0 = (R_1^0)^*..(R_k^0)^* \overset{\text{QR}}{=} Q_k^1\,R_k^1..R_1^1, \qquad \Sigma^j = (R_1^j)^*..(R_k^j)^* = \begin{pmatrix} \sigma_1^j & 0 & 0 \\ \cdot & \sigma_2^j & 0 \\ \cdot & \cdot & \sigma_3^j \end{pmatrix}$$

$$\Sigma^1 = (Q_k^0)^*\,\Phi(k\tau,u_0)\,Q_k^1 = (R_1^1)^*..(R_k^1)^* \overset{\text{QR}}{=} Q_k^2R_k^2..R_1^2,$$

$$\sigma_i^j = R_1^j[i,i]..R_k^j[i,i] \underset{j\to\infty}{\longrightarrow} \sigma_i(k\tau,u_0)$$

$\ldots$

$$\mathrm{LE}_i(T,u_0) \approx \mathrm{LE}_i^j(k\tau,u_0) = \frac{1}{t}\ln\sigma_i^j = \frac{1}{k\tau}\sum_{l=1}^k \ln R_l^j[i,i], \ \ \mathrm{LCE}_i(k\tau,u_0) \approx \mathrm{LE}_i^0(k\tau,u_0)$$

Lyapunov exponents computation: example

$$R(t)=\begin{pmatrix}1 & g(t)-g^{-1}(t)\\ 0 & 1\end{pmatrix},$$

$$g(t):=\exp(\tfrac{t}{10})$$

j	$\mathrm{LE}^j_{1,2}(5)$	$\mathrm{LE}^j_{1,2}(25)$	$\mathrm{LE}^j_{1,2}(100)$
0	0	0	0
1	±0.00797875	±0.04394912	±0.09360078
2	±0.01585661	±0.07379280	±0.09986978
3	±0.02353772	±0.08902280	±0.09999751
4	±0.03093577	±0.09563887	±0.09999995
5	±0.03797757	±0.09830568	±0.09999999
10	±0.06638388	±0.09998593	±0.10000000
50	±0.09993286	±0.09999999	±0.10000000
100	±0.09999998	±0.09999999	±0.10000000

Exact limit values:

$$\mathrm{LCE}_1=\lim_{t\to+\infty}\tfrac{1}{t}\ln g(t)=0.1,$$

$$\mathrm{LCE}_2=0,$$

$$\mathrm{LE}_{1,2}=\lim_{t\to+\infty}\tfrac{1}{t}\ln g^{\pm1}(t)=\pm0.1,$$

Finite-time values:

$$\mathrm{LCE}_1(t)=\tfrac{1}{t}\ln\Big(\big(g(t)-\tfrac{1}{g(t)}\big)^2+1\Big)^{\frac{1}{2}}\in(0,0.1],\ \mathrm{LCE}_2(t)\equiv0,\ \mathrm{LE}_{1,2}(t)\equiv\mathrm{LE}_{1,2}=\pm0.1$$

Approximation of LCEs by Benettin et al. (1980) becomes worse with $t\nearrow$:

$$\mathrm{LCE}_1(t)\underset{t\to+0}{\longrightarrow}0\equiv\mathrm{LE}^0_i(t)=\tfrac{1}{t}\ln1\equiv0,\quad \mathrm{LCE}_1(t)\underset{t\to+\infty}{\longrightarrow}0.1\neq\mathrm{LE}^0_i(t)=\tfrac{1}{t}\ln1\equiv0.$$

Relying on ergodicity the notions of LCEs and LEs often do not differ (see, e.g. **Eckmann & Ruelle (1985)**, **Wolf et al. (1985)**, and **Abarbanel et al.(1993)**), but **in a general case, the computations of LCEs by $\{\mathrm{LE}^j_i\}_{i=1}^n$ and LEs by $\{\mathrm{LE}^0_i\}_{i=1}^n$ may give non relevant results**.

Lyapunov dimension: ergodicity vs multistability

Can one expect the same LEs for almost all points? Multistability!

$(\{\varphi^t\}_{t\geq 0},(\mathbb{R}^n,||\cdot||))$, $D\varphi^t(u)$ singular val: $\sigma_1(t,u)\geq..\geq\sigma_n(t,u)>0$, $\mathrm{LE}_i(t,u)=\frac{1}{t}\ln\sigma_i(t,u)$

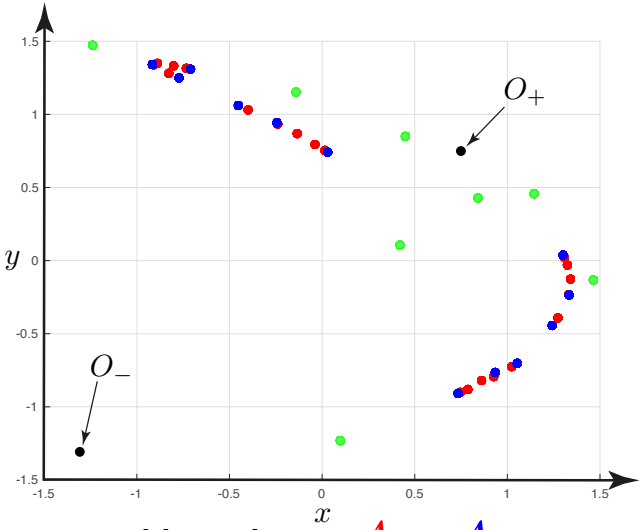
Oseledec, 1968 (ergodicity): $\lim\limits_{t\to+\infty}\mathrm{LE}_i(t,u)=\mathrm{LE}_i(u)\equiv\mathrm{LE}_i$ for μ-almost all u

Kaplan-Yorke, 1979: Lyapunov dimension $d_{\mathrm{L}}^{\mathrm{KY}}=j+\frac{\mathrm{LE}_1+\cdots+\mathrm{LE}_j}{|\mathrm{LE}_{j+1}|}$

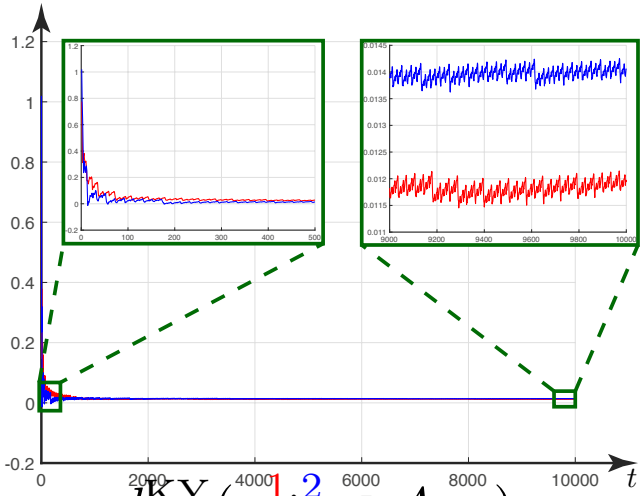
$j=\max\{m:\sum_1^m\mathrm{LE}_i\geq 0\}$, $\sum_1^{m+1}\mathrm{LE}_i<0$; absolute $\mathrm{LE}_s(u)\equiv\mathrm{LE}_s\ \forall u\in\mathbb{R}^n$

Henon map: $\varphi(x,\,y)=\left(1+y-ax^2,\,bx\right)$ multistability and coexistence of two self-excited attractor

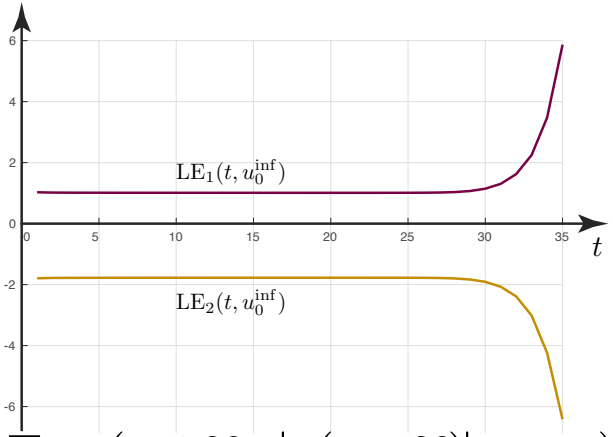
parameters $a=0.97, b=0.466$



attractors A_1, A_2



$d_{\mathrm{L}}^{\mathrm{KY}}(u_0^{1,2}\in A_{1,2})$



$\mathrm{LE}_{1,2}(t,u_0^\infty\colon |u(t,u_0^\infty)|\to\infty)$

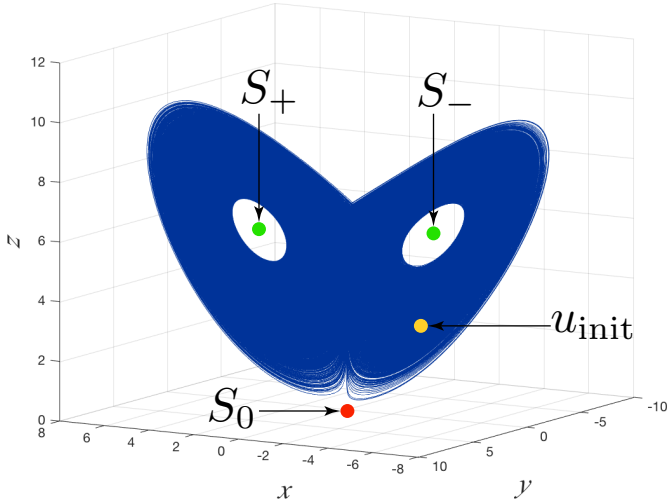
$\mathrm{LE}_1(u_0^1)\approx 0.011$, $d_{\mathrm{L}}^{\mathrm{KY}}(u_0^1)\approx 1.015$; $\mathrm{LE}_1(u_0^2)\approx 0.014$, $d_{\mathrm{L}}^{\mathrm{KY}}(u_0^2)\approx 1.018$, $d_{\mathrm{L}}^{\mathrm{KY}}(O_-)=1.569$, $d_{\mathrm{L}}^{\mathrm{KY}}(O_+)=1.427$

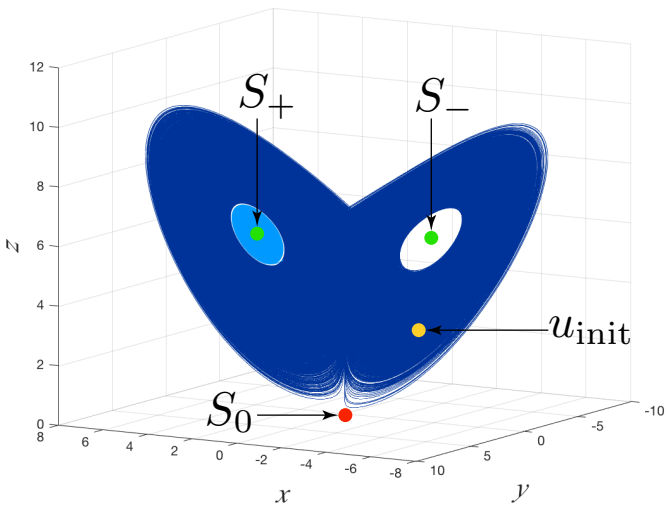
Kuznetsov N., Leonov G., Mokaev T., Finite-time and exact Lyapunov dimension of the Henon map, arXiv 1712.01270

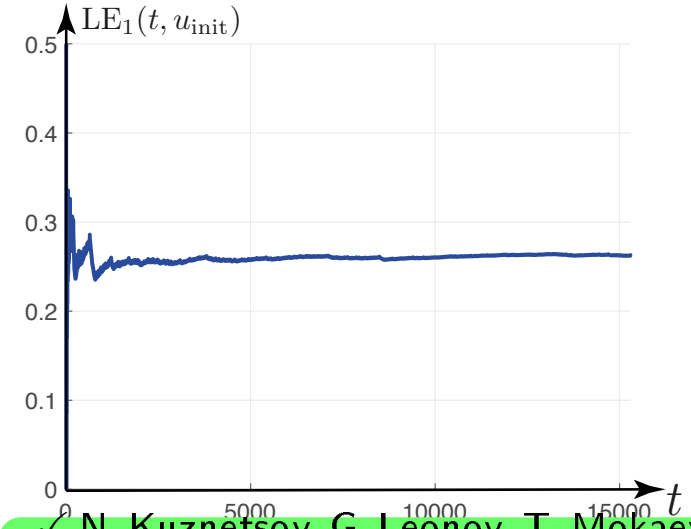
Lyapunov dimension: finite-time and limit values

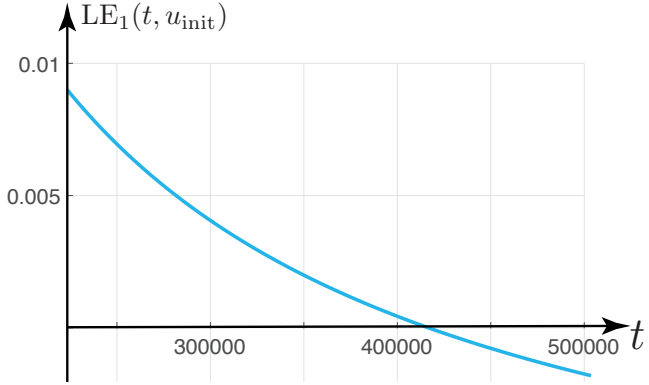
Can one compute the LEs limit value for a point? Long-time transient behavior!

$$\dot{x}=-\sigma(x-y)-ayz,\ \dot{y}=rx-y-xz,\ \dot{z}=-bz+xy,\ \sigma=-ar,\ r=6.485,\ a=-0.5,\ b=0.85$$









✓ N. Kuznetsov, G. Leonov, T. Mokaev, A. Prasad, M. Shrimali, Finite-time Lyapunov dimension and hidden attractor of the Rabinovich system, **Nonlinear Dynamics**, **2018** http://doi.org/10.1007/s11071-018-4054-z

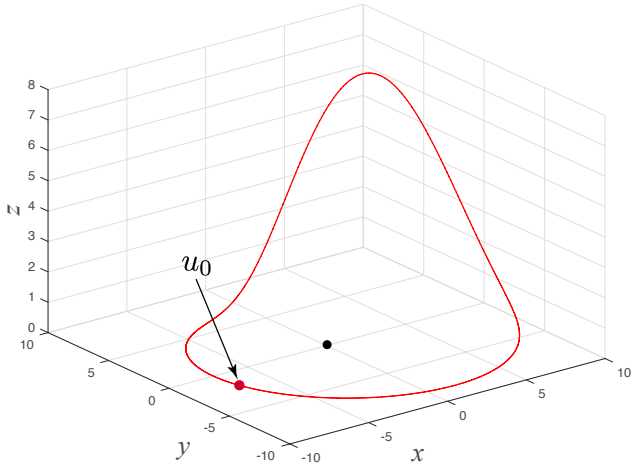
Lyapunov dimension: ergodicity vs shadowing

Can one expect the same LEs for all points of attractor? Embedded UPO!

Can one compute the LEs limit values for a given point? Finite-time shadowability!

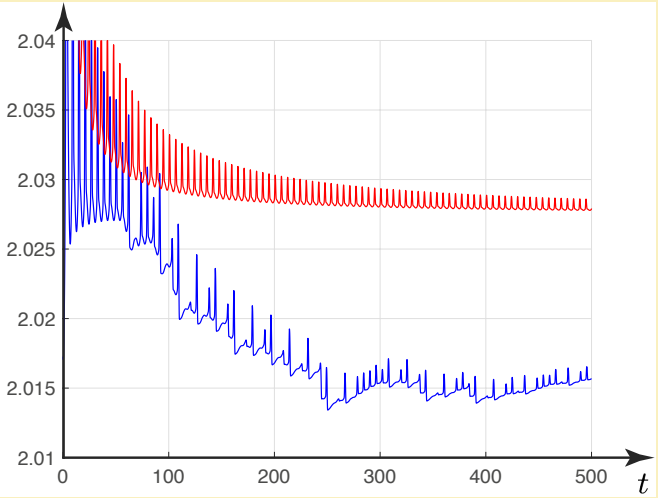
Rössler system: $\dot{x}=-y-z;\ \dot{y}=x+ay;\ \dot{z}=b-cz+xz,\ a=b=0.2,\ c=5.7.$

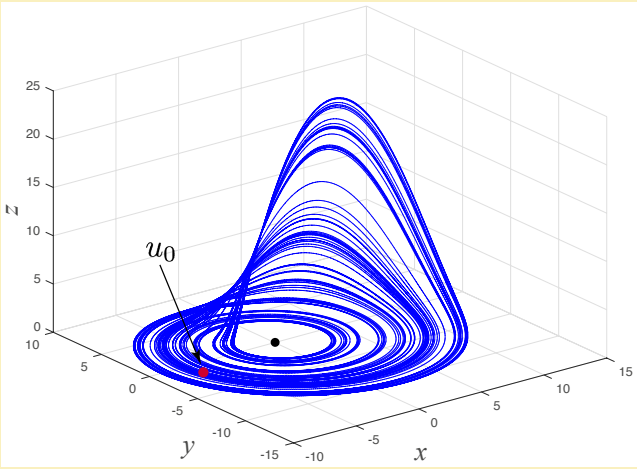
Pyragas UPO stabilization: $\dot{u}=f(u)+G\left[u(t-\tau)-u(t)\right]$, τ – period, G – gain



UPO, $\tau\approx5.88$, $G=0.3$, $u(t)$, $t\in[0,500]$

Lyapunov dimension





chaotic attractor, $G=0$, $u(t)$, $t\in[0,500]$

$$u_0=(-6.954527608751571,-1.957934292931243,0.015672234932247)$$

✓ Leonov G.A., Schumafov M.M., Kuznetsov N.V., Delayed feedback stabilization and the Huijberts-Michiels-Nijmeijer problem, **Differential equations**, 52(13), **2016**, 1707-1731

✓ Kuznetsov N., Leonov G., Shumafov M., A short survey on Pyragas time-delay feedback stabilization and odd number limitation, **IFAC-PapersOnLine**, 48(11), **2015**, 706-709

Finite-time Lyapunov dimension numerical approximation

Rabinovich system (interaction b/w waves in plasma):

$\dot{x}=-\sigma(x-y)-ayz;\ \dot{y}=rx-y-xz;\ \dot{z}=-bz+xy$

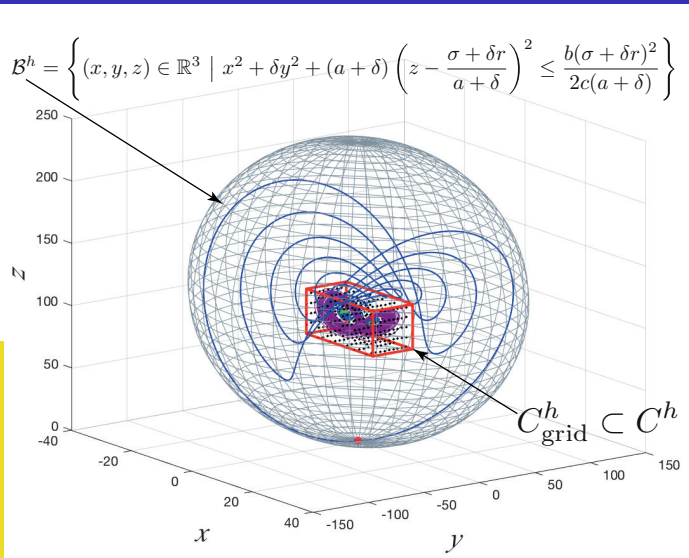
Hidden attractor: $r=100,\ a=-9.965\cdot 10^{-3},$

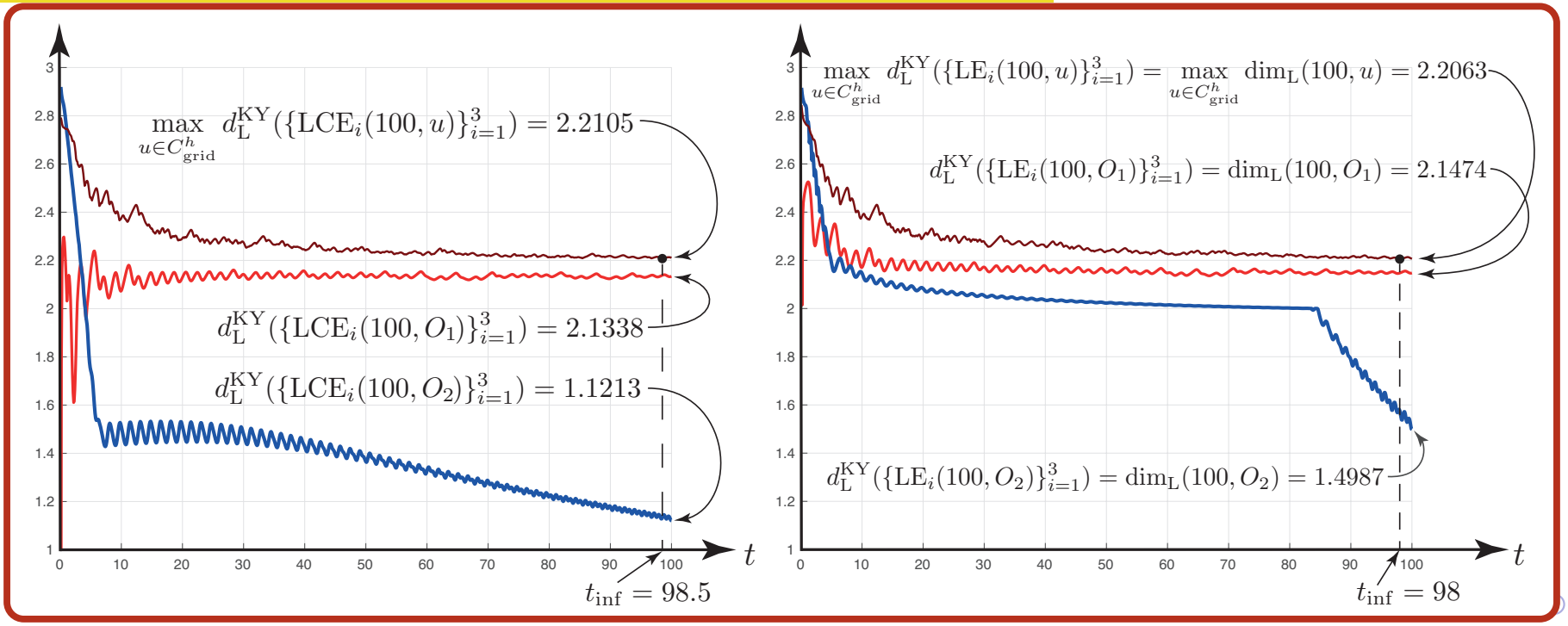
$b=7.7454\cdot 10^{-2},\ \sigma=-ar.$

Grid: $C^h_{\text{grid}}=[-11:5:11]\times[-17:5:19]\times[80:5:117]$

$$\dim_{\mathrm{H}} K \le \dim_{\mathrm{L}} K \le \inf_{t\in[0,100]} \max_{u\in C^h_{\text{grid}}} d_{\mathrm{L}}^{\mathrm{KY}}(\{\mathrm{LE}_i(t,u)\}_{i=1}^3)$$

$$\le \max_{u\in C^h_{\text{grid}}} d_{\mathrm{L}}^{\mathrm{KY}}(\{\mathrm{LE}_i(100,u)\}_{i=1}^3)$$





Content

Homoclinic bifurcations, Tricomi problem

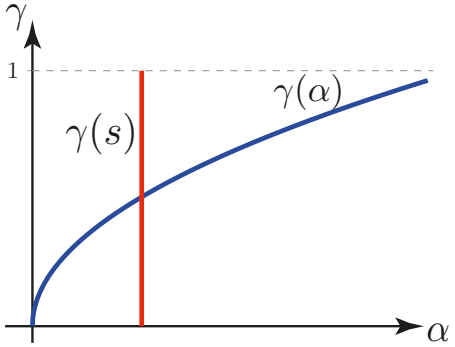
F. Tricomi, **1933**, Integrazione di un'equazione differenziale presentatasi in electrotechnica.

Pendulum motion:

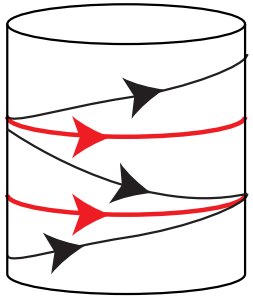
$$\ddot{\theta} + \alpha\dot{\theta} + \sin\theta = \gamma,$$

$$(\alpha > 0,\ \gamma < 1)$$

$\gamma(\alpha)$ — **homoclinic bifurcation**.



$\gamma(0) = 1, \gamma(1) = 0$



stable & unstable cycles of 2nd kind

ODE : $\dot{\mathrm{x}} = f(\mathrm{x}, q), \quad \mathrm{x} \in \mathbb{R}^n, \quad q \in \mathbb{R}^m.$

$f(\mathrm{x}, q)$ — smooth vector-function, $\{\mathrm{x}\}$ — phase space, $\{q\}$ — space of parameters.

Homoclinic orbit : $\mathrm{x}(t) \quad : \quad \lim\limits_{t\to+\infty} \mathrm{x}(t) = \lim\limits_{t\to-\infty} \mathrm{x}(t) = \mathrm{x}_0, \quad \mathrm{x}_0$ – equilibrium.

Smooth path : $\gamma(s) \in \{q\}, \quad s \in [0, 1].$

Tricomi Problem: Is there a point $q_0 \in \gamma(s)$: q_0 – point of homoclinic bifurcation?

✓ **G.A. Leonov**, The Tricomi problem on the existence of homoclinic orbits in dissipative systems. **J. Appl. Math. Mech.**, 77(3), **2013**

Fishing principle

Let: ✓ $\mathrm{x}(t,s)^+$ – outgoing separatrix of saddle x_0 with a 1D unstable manifold;
✓ $\mathrm{x}_\Omega(s)^+$ – point of the first crossing of separatrix $\mathrm{x}(t,s)^+$ with the closed set Ω;
✓ if no such crossing : assume $\mathrm{x}_\Omega(s)^+ = \emptyset$;

Fishing Principle (Leonov, 2012): **Suppose for $\gamma(s)$ there is $(n-1)$-dim. bounded manifold Ω with a piecewise-smooth edge $\partial\Omega$ possessing properties:**

(i) for any $\mathrm{x} \in \Omega \setminus \partial\Omega$ and $s \in [0,1]$, the vector $f(\mathrm{x},\gamma(s))$ is transversal to the manifold $\Omega \setminus \partial\Omega$;

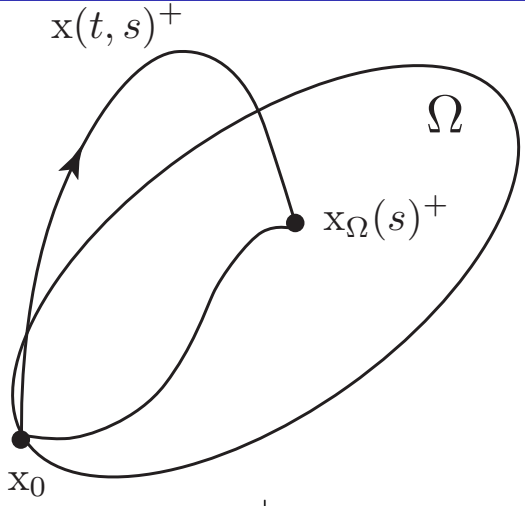
(ii) for any $s \in [0,1]$, $f(\mathrm{x}_0,\gamma(s)) = 0$, the point $\mathrm{x}_0 \in \partial\Omega$ is a saddle;

(iii) for $s = 0$ the inclusion $\mathrm{x}_\Omega(0)^+ \in \Omega \setminus \partial\Omega$ is valid;

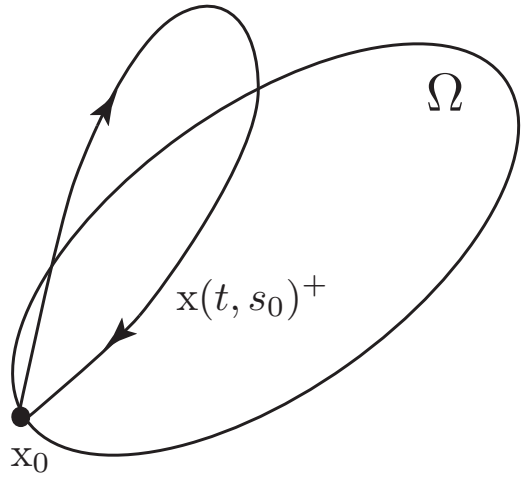
(iv) for $s = 1$ the relation $\mathrm{x}_\Omega(1)^+ = \emptyset$ is valid;

(v) $\forall\ s \in [0,1]$ and $\mathrm{y} \in \partial\Omega \setminus \mathrm{x}_0$ there exists neiborhood $U(\mathrm{y},\delta) = \{\mathrm{x} \,|\, |\mathrm{x}-\mathrm{y}| < \delta\}$: $\mathrm{x}_\Omega(s)^+ \notin U(\mathrm{y},\delta)$.

If (i)–(v) satisfied $\Rightarrow \exists\ s_0 \in [0,1]$: $\mathrm{x}(t,s_0)^+$ is a homoclinic trajectory of the saddle point x_0.



$\mathrm{x}(t,s)^+,\ s \in [0,s_0)$



$\mathrm{x}(t,s)^+,\ s = s_0$

✓ **G.A. Leonov**, General existence conditions of homoclinic trajectories in dissipative systems. Lorenz, Shimizu-Morioka, Lu and Chen systems, **Phys Lett A**, 376(45), **2012**, 3045-3050

Homoclinic orbit in the Lorenz system

G.A. Leonov, Estimation of loop-bifurcation parameters for a saddle-point separatrix of a Lorenz system, Differential Equations, 24(6), **1988**

G.A. Leonov, Bounds for attractors and the existence of homoclinic orbits in the Lorenz system, Journal of Applied Mathematics and Mechanics, 65(1), **2001**

G.A. Leonov, General existence conditions of homoclinic trajectories in dissipative systems. Phys. Lett. A 376, **2012**

Lorenz system:

$$\dot{x} = -\sigma x + \sigma y,$$
$$\dot{y} = rx - y - xz,$$
$$\dot{z} = -bz + xy,$$

$r, \sigma, b > 0$ – parameters.

Thm. For σ and b fixed, there exists $r \in (1, +\infty)$, corresponding to the homoclinic trajectory of the saddle $x = y = z = 0$, iff $2b + 1 < 3\sigma$.

V.N. Belykh (1984), X. Chen (1996), S.P. Hastings, W.C. Troy (1996)

Criteria of homoclinic orbit existence

2012-till now: Lorenz system, Shimizu-Morioka system, Chen system, Lu system, Tigan system, Glukhovsky-Dolzhansky system, and others

✓ Leonov G.A., Kuznetsov N.V., Mokaev T.N., Homoclinic orbits, and self-excited and hidden attractors in a Lorenz-like system describing convective fluid motion, **European Physical Journal Special Topics**, 224, **2015**, 1421-1458

✓ Leonov G.A., Kuznetsov N.V., On differences and similarities in the analysis of Lorenz, Chen and Lu systems, **Applied Mathematics and Computation**, 256, **2015**, 334-343

✓ Леонов, Г.А., Задача Трикоми для динамической системы Шимицу-Мориока, **Доклады Академии наук**, 447, **2012**

✓ Leonov, G.A., General existence conditions of homoclinic trajectories in dissipative systems. Lorenz, Shimizu-Morioka, Lu and Chen systems, **Phys. Lett. A**, 376, **2012**

✓ Leonov, G.A., Shilnikov chaos in Lorenz-like systems, **Int. J. Bifurc. Chaos**, 23(3), **2013**

✓ Leonov, G.A.: The Tricomi problem on the existence of homoclinic orbits in dissipative systems. **J. Appl. Math. Mech.**, 77(3), **2013**

✓ Leonov, G.A., Fishing principle for homoclinic and heteroclinic trajectories, **Nonlinear Dyn.**, 78(4), **2014**

✓ Leonov G.A., Existence conditions of homoclinic trajectories in Tigan system, **Int. J. Bifurc. Chaos**, 25(13), **2015**, 1550175

✓ Леонов, Г.А., Каскад бифуркаций в системах лоренцовского типа: рождение странного аттрактора, бифуркация катастрофы голубого неба и девяти гомоклинических бифуркаций, **Доклады Академии наук**, 464(4), **2015**, 391-395

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Homoclinic Bifurcations in Lorenz-like systems

Lorenz-like system:

$$\begin{cases} \dot{x} = v, \\ \dot{v} = -\lambda v - xu + x - x^3, \\ \dot{u} = -\alpha u - \beta x v, \end{cases}$$

α, β, λ – real numbers, $\alpha > 0$.

includes:

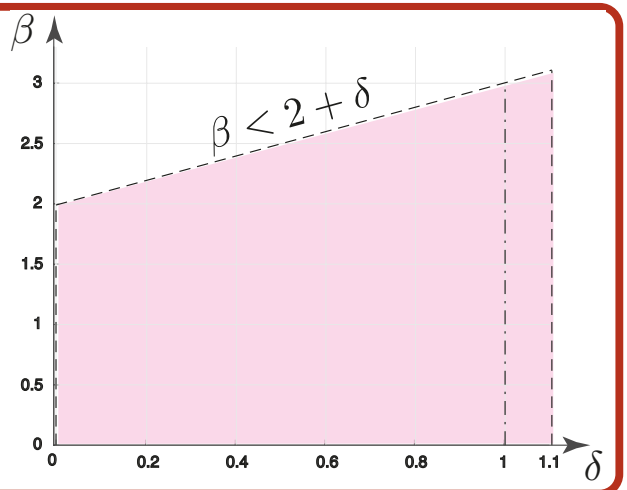
- ✓ **Lorenz system**
- ✓ **Shimizu-Morioka system**
- ✓ **Chen system**
- ✓ **Lu system**
- ✓ **Tigan system**

Thm. [Leonov,Mokaev,2017] Consider a smooth path $\lambda(s)$, $\alpha(s)$, $\beta(s)$, $s \in [0,\ 1)$.
Let $\lambda(0) = 0, \lim\limits_{s\to 1} \lambda(s) = +\infty, \limsup\limits_{s\to 1} \alpha(s) < +\infty, \limsup\limits_{s\to 1} \beta(s) < +\infty$ and the following condition holds: $\alpha(s)(\sqrt{\lambda(s)^2 + 4} + \lambda(s)) > 2(\beta(s) - 2),\ \ \forall s \in [0, 1]$. Then there exists $s_0 \in (0, 1)$: the Lorenz-like system with $\alpha(s_0)$, $\beta(s_0)$, $\lambda(s_0)$ **has a homoclinic trajectory**.

Consider path: $\lambda(s) = \frac{s}{\sqrt{1-s}}, \quad \alpha(s) = \delta\sqrt{1-s},$
$\beta(s) \in (0,\ 2+\delta),\ \ \delta > 0,\ \ s \in [0,\ 1)\ \ (\star)$

Corollary. There exists $s_0 \in (0,\ 1)$ such that Lorenz-like system with $s = s_0$ and parameters $(\star)$ has a homoclinic orbit.

Saddle value: ✓ positive, if $\delta < 1$;
✓ zero, if $\delta = 1$; ✓ negative, if $\delta > 1$;



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Questions

Thank you for your attention!

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