Acoustic turbulence: from Zakharov-Sagdeev spectrum to Kadomtsev-Petvishvili spectrum

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OUTLINE

- Introduction. Acoustic turbulence: weak vs. strong.
- Basic equations
- 3D weak turbulence
- Numerical simulations: weak wave dispersion
- Dispersionless acoustic turbulence
- Conclusion

Key references

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Introduction. Acoustic turbulence: weak vs. strong

- As well known, the weak turbulence theory (WTT) is based on the assumption that the nonlinear interaction of waves is weak in comparison with the linear wave dispersion. Thereby, initially Gaussian-distributed linear waves with different $\mathbf{k}$ almost remain this property when weak nonlinearity is taken into account. Each wave moving with its own frequency and wave vector experiences the influence of other waves at distances L greater its wavelength, $\propto k^{-1}$. This leads to weak turbulence description using kinetic equations for n_k .
- In the leading approximation in nonlinearity, the kinetic equations for WTT describe either decay processes ($1 \to 2$),

$$\omega_k = \omega_{k_1} + \omega_{k_2}, \quad \mathbf{k} = \mathbf{k}_1 + \mathbf{k}_2,$$

Introduction. Acoustic turbulence: weak vs. strong

- or 4-wave interaction with the resonance condition

$$\omega_k + \omega_{k_1} = \omega_{k_2} + \omega_{k_3}, \quad \mathbf{k} + \mathbf{k}_1 = \mathbf{k}_2 + \mathbf{k}_3.$$

- For linear dependence of ω on $\mathbf{k}$ these conditions are satisfied for parallel $\mathbf{k_i}$ For weak positive dispersion,

$$\omega_k = kc_s(1 + a^2k^2 + ...),$$

 the decay processes are allowed. For negative dispersion, the scattering $2 \to 2$ is allowed.

Introduction. Acoustic turbulence: weak vs. strong

- In this lecture, we consider both weak positive dispersion and dispersionless cases. In the WTT, the kinetic equation is written as

$$\frac{\partial n_k}{\partial t} = 2\pi \int d\mathbf{k}_1 d\mathbf{k}_2 \left(T_{kk_1k_2} - T_{k_1kk_2} - T_{k_2kk_1}\right),$$

where

$$T_{kk_1k_2} = |V_{kk_1k_2}|^2 (n_{k_1}n_{k_2} - n_k n_{k_2} - n_k n_{k_1})$$

$$\delta\left(\mathbf{k} - \mathbf{k}_1 - \mathbf{k}_2\right) \delta\left(\omega_k - \omega_{k_1} - \omega_{k_2}\right).$$

- Turbulence spectrum in the 3D isotropic case is given as $E(k) = (2\pi^2)^{-1} k^3 c_s n(k)$.

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Introduction. Acoustic turbulence: weak vs. strong

- In the weak dispersion limit,

$$V_{kk_1k_2} = C(kk_1k_2)^{1/2},$$

is the homogeneous function of their arguments of power $3/2$.

- Therefore in the long-wave region the kinetic equation has the power type spectrum as it was demonstrated first time by Zakharov (1965) and later by Zakharov and Sagdeev (1970):

$$E(k) = C_{KZ}\epsilon^{1/2}k^{-3/2}.$$

This is the Kolmogorov spectrum corresponding to constant energy flux ϵ.

- According to our calculation,

$$C_{KZ} = \left[\frac{3}{4\pi(\pi - 1 + \ln 16)}\right]^{1/2} \approx 0.22.$$

Introduction. Acoustic turbulence: weak vs. strong

- As was shown by Zakharov (1965), in spite of presence of singularities provided by two resonant conditions, after averaging over angles in the KE singularities occur integrable. This is a property of 3D when the spectrum can be found by means of Zakharov transformation. In 2D, the spectrum (see Griffin, Krstulovic, L'vov, Nazarenko (2022)) contains the dispersion length: $E(k) \propto a^{-1}k^{-1}$.
- The existence of the Zakharov-Sagdeev spectrum was confirmed in a number of papers mainly for 3D isotropic KE.
- In this lecture, we will show that in direct numerical simulation of 3D acoustic turbulence the structure of the spectra is not isotropic, especially in the region of small k.

Introduction. Acoustic turbulence: weak vs. strong

- In 3D simulations we first time observed the appearance of the WT Zakharov-Sagdeev (ZS) spectrum. This spectrum is realized at large enough k where the turbulence is almost isotropic. In the small k, close to pumping, the distribution is very anisotropic, representing set of jets.
- Unlike WTT, in the dispersionless case, the situation is very different. According to Kadomtsev-Petviashvili (1972) the (strong) acoustic turbulence can be considered as a random set of shocks which provides the KP spectrum $E(k) \propto k^{-2}$. Such dependence appears due to the density jumps.

Introduction. Acoustic turbulence: weak vs. strong

- As we will show in this in our numerical experiments (at $a = 0$) the KP spectrum is realized for large enough pumping. The main contribution to the spectrum comes from the shocks.
- If the pumping amplitudes have intermediate values, instead of the KP spectrum we observed ZS spectrum. The latter is connected with the jet distributions in k-space. These jets start at the pumping region and vanish at the large k. The jets have the form of cones.

Introduction. Acoustic turbulence: weak vs. strong

- Appearance of ZS spectrum for the intermediate pumping in the dispersionless case is connected with compensation of nonlinearity by diffraction for each jet. Diffraction in this case plays a role of dispersion. This compensation gives the inverse characteristic time $\propto n_k$ that leads to the ZS spectrum.
- In the WT regime, the jets concentrate near pumping region, have the cone forms. With increasing k the cone angles grow and the energy distribution comes almost isotropic where ZS spectrum is formed.

Basic equations

- DNS of acoustic turbulence was carried out in the framework of the nonlinear string equation (Zakharov, 1965):

$$u_{tt} = \Delta u - 2a^2\Delta^2 u + \Delta(u^2).$$

In 1D, this equation is integrable by IST (Zakharov 1973). In 3D, this model was first used by Zakharov to study WT. The dispersion law is

$$\omega^2 = k^2 + 2a^2k^4, \qquad k = |\mathbf{k}|,$$

which at $ka \ll 1$ gives weak positive dispersion.

Basic equations

- The equation belongs to Hamiltonian systems:

$$u_t = \frac{\delta H}{\delta \phi}, \qquad \phi_t = -\frac{\delta H}{\delta u},$$

where u has the meaning of density fluctuation, ϕ is the hydrodynamic potential ($\mathbf{v} = \nabla \phi$), and

$$H = \frac{1}{2}\int \left[(\nabla\phi)^2 + u^2\right] d\mathbf{r} + \int a^2(\nabla u)^2 d\mathbf{r} + \frac{1}{3}\int u^3 d\mathbf{r}$$

$$\equiv H_1 + H_2 + H_3.$$

- Here H_1 is the sum of the kinetic and potential energies of linear dispersionless waves. H_2 is responsible for the dispersive part , and H_3 describes the nonlinearity.

Basic equations

- Transformation to the normal variables a_k and a_k^* is written as

$$u_k = \left(\frac{k^2}{2\omega_k}\right)^{1/2} (a_k + a_{-k}^*),$$
$$\phi_k = -i\left(\frac{\omega_k}{2k^2}\right)^{1/2} (a_k - a_{-k}^*),$$

when equations take the standard form :

$$\frac{\partial a_k}{\partial t} = -i\delta H/\delta a_k^*,$$

where

$$H = \int \omega_k |a_k|^2 d\mathbf{k} + \frac{1}{2}\int V_{k_1k_2k_3}\left(a_{k_1}^* a_{k_2} a_{k_3} + \right.$$

$$\left. + a_{k_1} a_{k_2}^* a_{k_3}^*\right) \delta\left(\mathbf{k}_1 - \mathbf{k}_2 - \mathbf{k}_3\right) d\mathbf{k}_1 d\mathbf{k}_2 d\mathbf{k}_3.$$

Basic equations

- At $ka \ll 1$ we will take into account the dispersion only in the quadratic H,
$$\omega_k = k(1 + a^2 k^2),$$
but in the matrix element, it will be neglected:
$$V_{k_1 k_2 k_3} = \frac{1}{8\pi^{3/2}} \left(k_1 k_2 k_3\right)^{1/2}.$$
- This limit in the WTT gives ZS spectrum as exact solution of the KE corresponding to the constant energy flux ϵ.
- Note that the WT regime realizes when between H_1, H_2 and H_3 the following inequalities are satisfied:
$$H_1 \gg H_2 \gg H_3$$
which were verified in the DNS.

Basic equations

- As was first noted by Zakharov, in the case of 3D isotropic distributions, the dispersion contribution in ω_k can be neglected, despite the presence of the product of two delta functions with respect to frequencies and wave vectors in the collision term giving a singularity in the kinetic equation. This singularity turns out to be integrable after averaging over the angles. As a result, the kinetic equation admits a stationary power-law solution: $n_k \propto k^{\alpha}$. The exponent α for the Kolmogorov-type spectrum is found using the Zakharov transformations: $\alpha = -9/2$, which corresponds to the Zakharov-Sagdeev spectrum:

$$E(k) = C_{KZ}\epsilon^{1/2}k^{-3/2}.$$

Basic equations

- For modeling turbulence we include both pumping and damping terms,

$$u_t = -\Delta\phi + \mathcal{F}(\mathbf{k}, t) - \gamma_k u, \;\; \phi_t = -u + 2a^2\Delta u - u^2,$$

- γ_k is responsible for dissipation and the forcing term $\mathcal{F}(\mathbf{k}, t)$ are given as:
  $\gamma_k = 0, \quad k \le k_d$,
  $\gamma_k = \gamma_0, \quad k > k_d$,
  $\mathcal{F}(\mathbf{k}, t) = F(k) \cdot \exp[iR(\mathbf{k}, t)]$,
  $F(k) = F_0 \cdot \exp[-(k - k_1)^4/k_2^4], \quad k \le k_2$,
  $F(k) = 0, \quad k > k_2$.

Basic equations

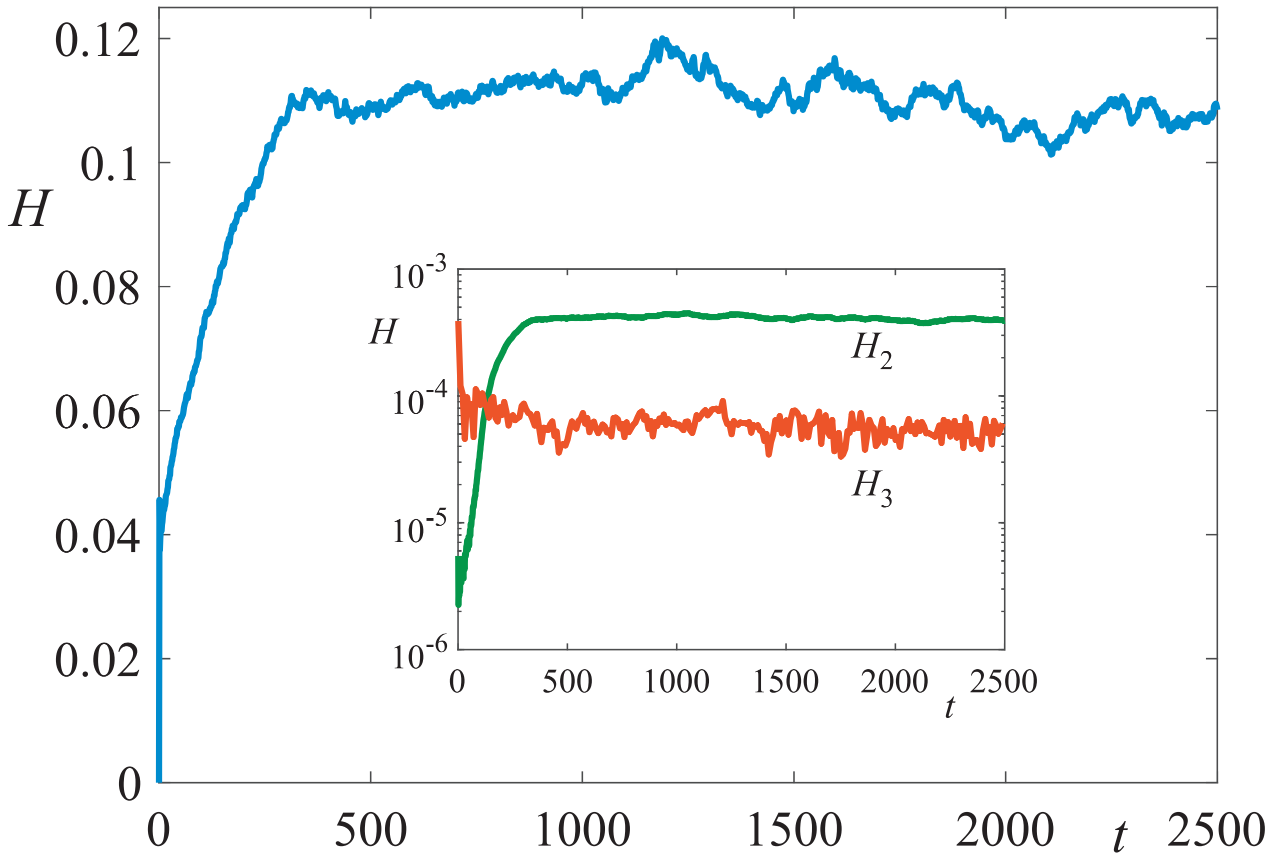
- Here $R(\mathbf{k},t)$ are random numbers uniformly distributed in the interval $[0, 2\pi]$, γ_0 and F_0 are constants. k_1 corresponds to the maximum pumping, k_2 sets its width, and $k_3$is the scale at which dissipation occurs.
- Numerical integration of the system was carried out in a periodic domain $(2\pi)^3$ using spectral methods with the total number of harmonics $N^3 = 512^3$. To suppress the aliasing effect we null harmonics with $k_a \geq N/3$. We present results of numerical simulation for the following parameters: $k_d = 125$, $k_1 = 3$, $k_2^4 = 6$, $\gamma_0 = 100$, $a = 2.5 \cdot 10^{-3}$, $F_0 = 5 \cdot 10^5$. With this choice of parameters, the inertial interval was more than one decade. The maximum dispersion addition at the end of the inertial interval, $k = k_d$, was $(k_d a)^2 \approx 0.1$.

Simulation results: WT

- Numerically we observed a transition to weak turbulence regime. Fig. 1 shows how the total energy of the system evolves. The rather quick transition can be seen to the quasi-stationary regime. The inset to Fig.1 shows the time dependencies of the dispersive part of the energy H_2 and the nonlinear interaction energy H_3. Both contributions H_2 and H_3 turn out to be small compared to H_1. The dispersive part of the energy exceeds the energy of the nonlinear interaction by almost an order of magnitude, which indicates on the realization of a weakly nonlinear regime. Thus, the total energy in the inertial interval is approximately equal to $H_1 \approx \int \epsilon_k d\mathbf{k}$, where $\epsilon_k = k|a_k|^2$ is the wave energy density in k-space.

Simulation results

Total energy of the system versus time for $a = 2.5 \cdot 10^{-3}$. The inset shows the time dependencies of the dispersive part of the energy H_2 and the nonlinear interaction energy H_3.

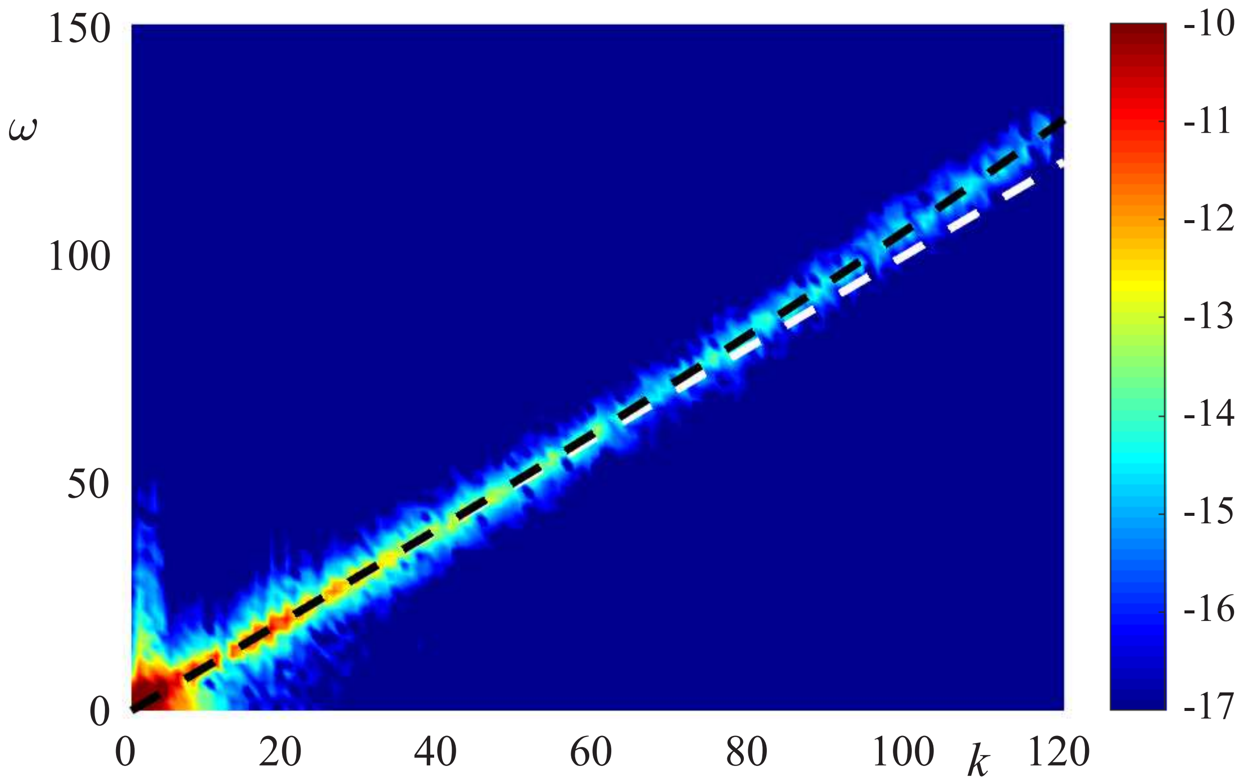


Simulation results: WT

- The behavior of the spectrum of space-time Fourier transform of the function $u(\mathbf{r}, t)$ shown in Fig.2 also testifies to the weakly nonlinear character of wave propagation. The figure shows that the wave energy is concentrated along the linear dispersion relation. Line broadening is due to nonlinearity. For almost the entire inertial interval, this broadening does not exceed the linear dispersion. For small k, the broadening is comparable to the dispersion. For larger k, the dispersion exceeds the nonlinear broadening, which agrees with the ratio of the corresponding contributions H_2 and H_3.

Simulation results: WT

The space-time Fourier transform $|u(\mathbf{k},\omega)|^2$ is shown in logarithmic scale. The black dotted line corresponds to the exact value of the dispersion curve, the white dotted line corresponds to the non-dispersive wave propagation, $\omega = |\mathbf{k}|$

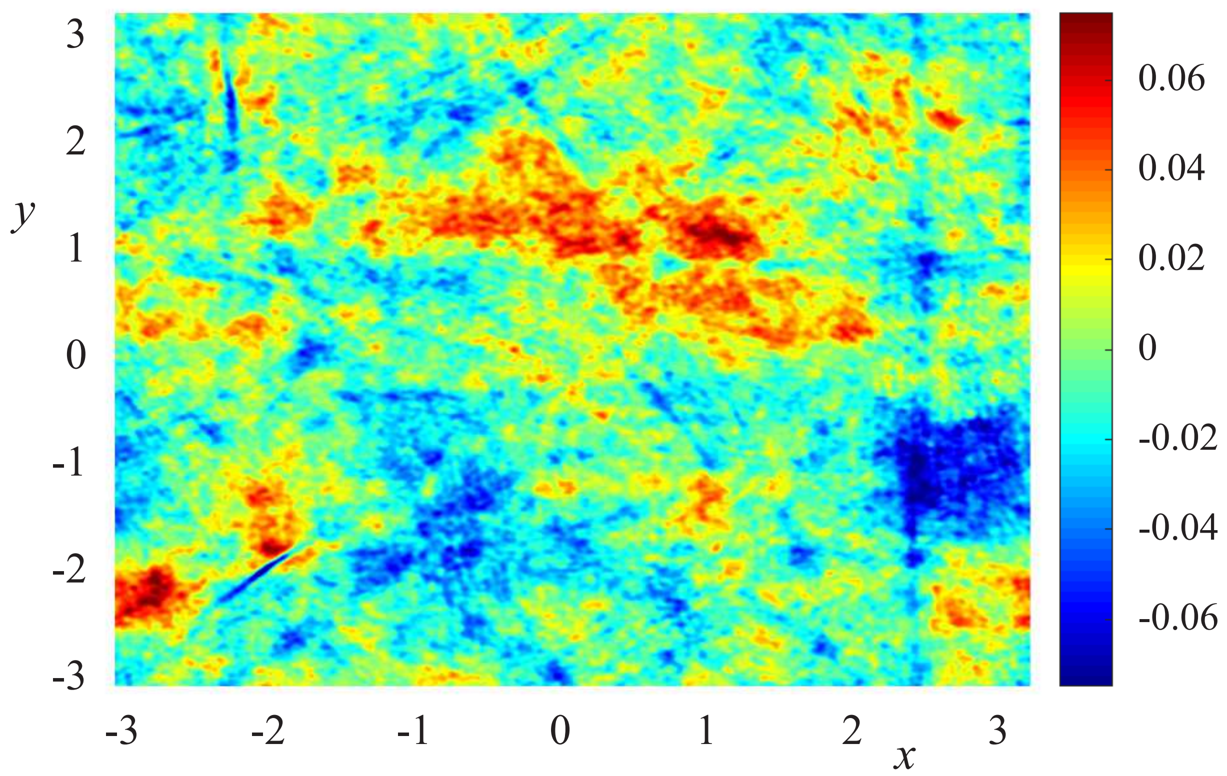


Simulation results: WT

- The numerical experiment shows that after the system enters the quasi-stationary state, the behavior of $u(\mathbf{r})$ acquires a complex (turbulent) character. In Fig.3 this behavior demonstrates the dependence of the function $u(\mathbf{r})$ in the $z = 0$ plane for the quasi-stationary state at the moment $t = 2500$. At the same time, the distribution of the energy density ϵ_k of turbulent fluctuations in the k-space is not isotropic. The anisotropy is especially pronounced in the region of small wavenumbers near the pumping.

Simulation results: WT

Section of the function $u(\mathbf{r})$ by $z = 0$ plane is shown at the moment $t = 2500$ corresponding to the quasi-stationary state.

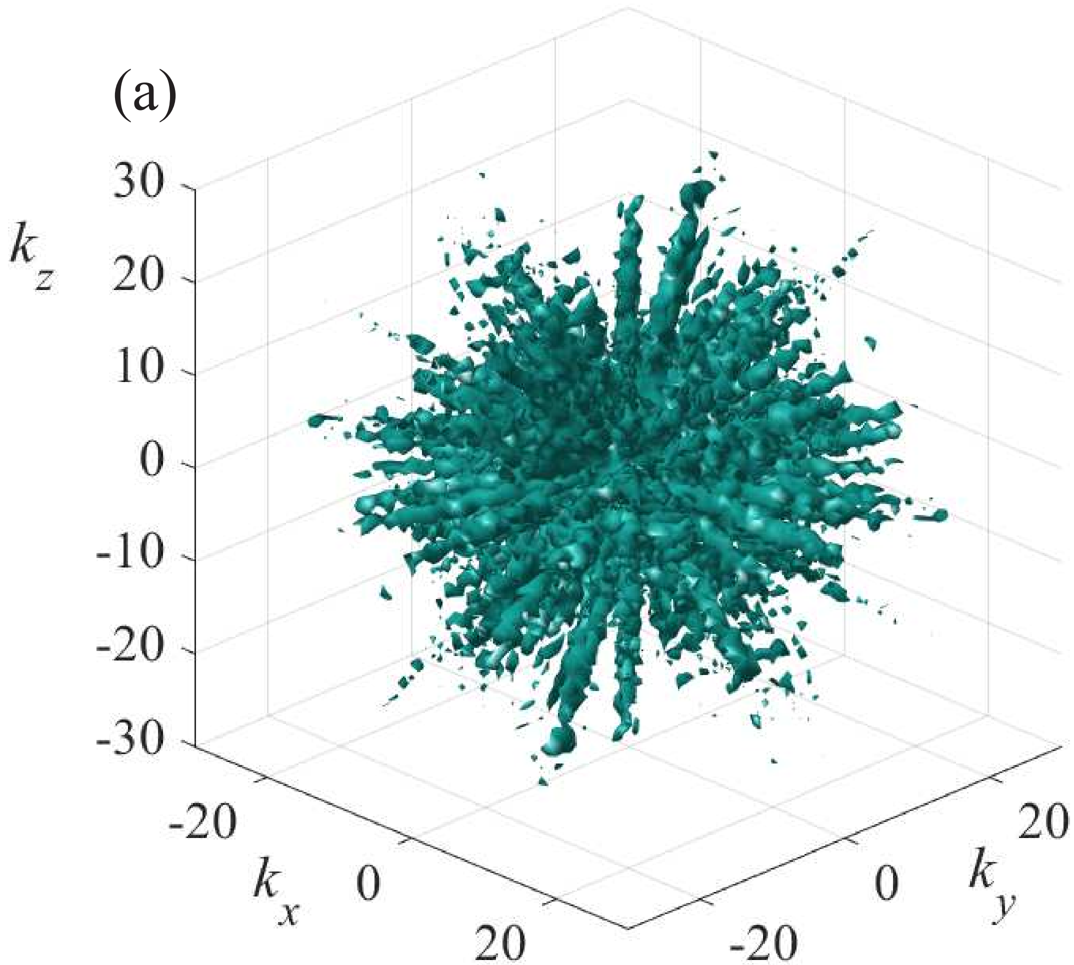


Simulation results: WT

- On Fig.4 we present three isosurfaces of the function $|u_{\mathbf{k}}|\ (= \epsilon_k^{1/2})$. As seen, in the region of small wavenumbers, structures with a large number of jets in the form of narrow cones appear in the distribution of turbulent fluctuations. The onset of such structures is the result of resonant wave interactions at very small k close to the pumping region when dispersion can be neglected. As k increases, the cones broaden and the distribution tends to be isotropic, see Fig.5. In this figure, the blue color (at $k \geq 30$) shows a tendency to spectrum isotropization, which is associated with an increase in dispersion with growing k and accordingly with an angular broadening of the resonant surface by an angle of the order of ka.

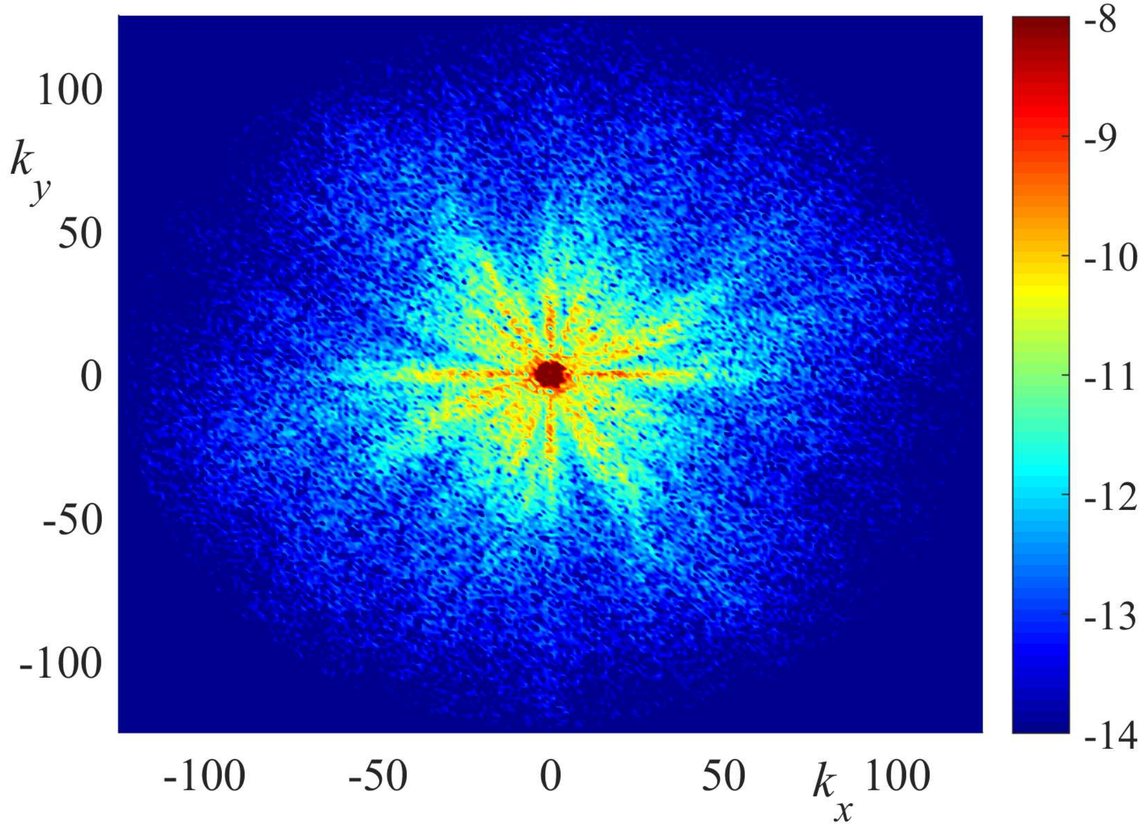
Simulation results: WT

Isosurfaces of the Fourier spectrum of $|u_{\mathbf{k}}| = u_0 = 5 \cdot 10^{-5}$, $t = 2500$.



Simulation results: WT

Fourier spectrum of $|u_{\mathbf{k}}| \equiv \epsilon_k^{1/2}$ in section $k_z = 0$ (logarithmic scale), $t = 2500$



Simulation results: WT

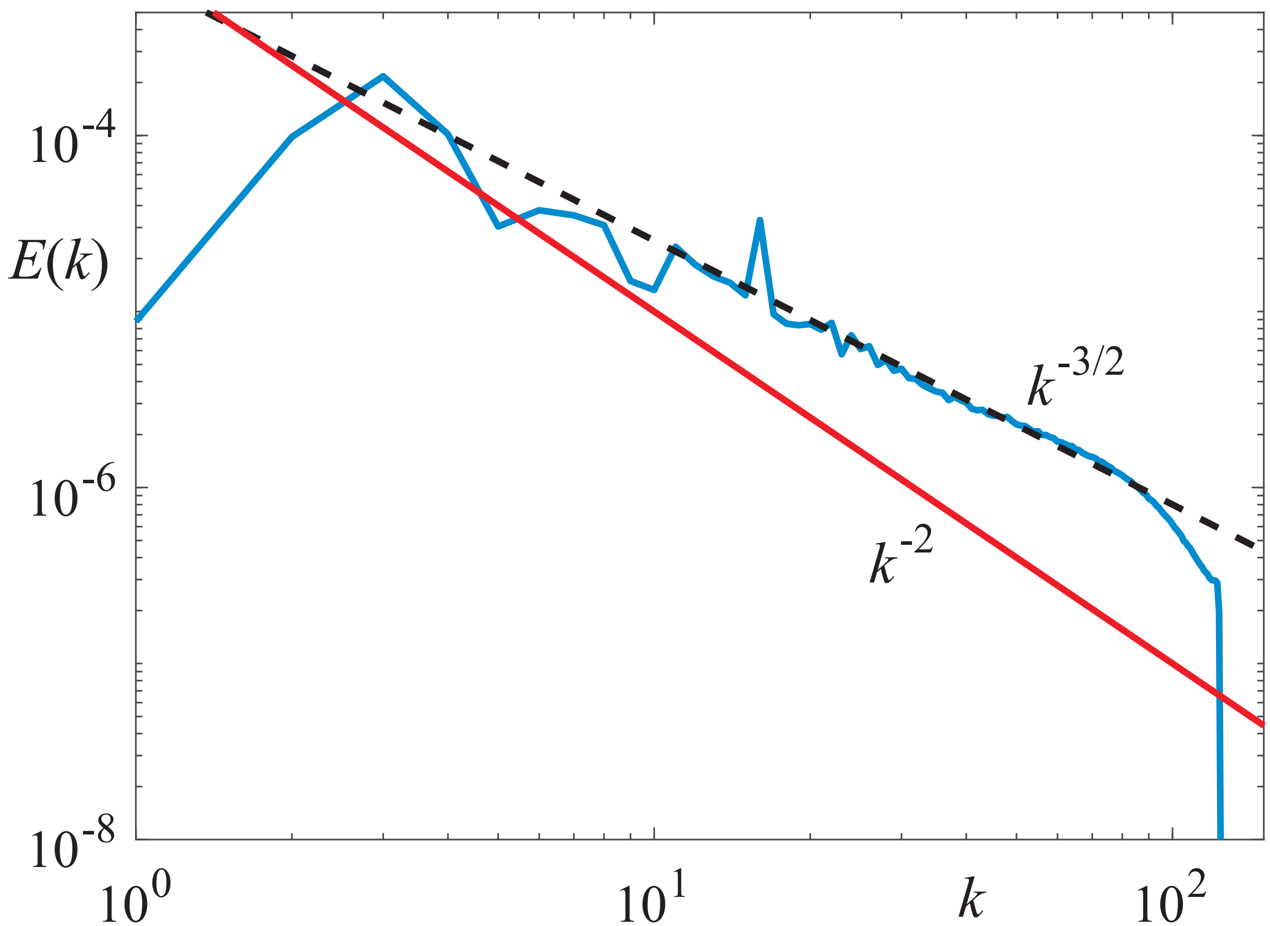
- The generation of jets is associated with two possible causes: linear and non-linear. The first one is the discreteness in the pumping region, $1 \leq k \leq 6$. Secondly, this is the tendency of the dispersion to zero at $k \to 0$; the 3-wave resonance conditions are satisfied for an arbitrary ray. The beams that form the jets have an advantage over other beams. This process, the cooperation of rays into a jet, has a clearly nonlinear character. This fact follows from the numerical simulations, for which the contrasts in intensity in the jets and the regions between them are significant: the difference reaches two orders of magnitude. Such a jump in intensity can not be explained only by a small anisotropy in pumping, but has a nonlinear origin, possibly due to the acoustic collapse.

Simulation results: WT

- It is clearly seen from Fig.6 that the spectrum of $E(k)$ acquires a power law behavior. There are two regions with different behavior of the spectrum in the inertial interval. In the region of large k, the spectrum of weak acoustic turbulence coincides with the ZS spectrum with high accuracy, and in the long-wave region, deviations from this spectrum are observed, which, in our opinion, arise due to jets, whose role is significant at small k. It should be noted that similar large deviations of an oscillatory nature from the Zakharov-Sagdeev spectrum were observed numerically in the framework of the GP equation, Proment and Co (2012). These deviations can be related to the anisotropy caused by the presence of jets. No jets were found in this experiment .

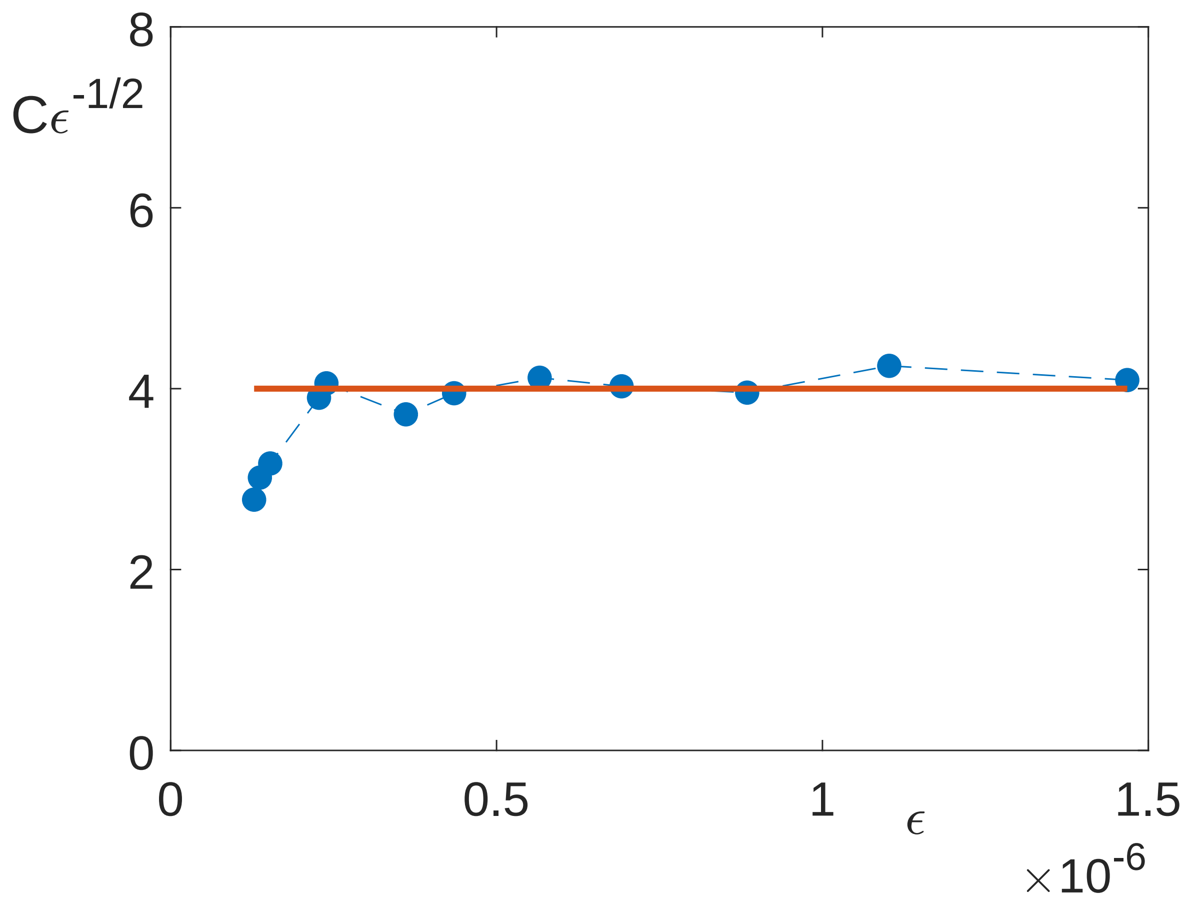
Simulation results: WT

The turbulence spectrum $E(k)$ measured in the quasi-stationary state, the black dotted line corresponds to the Zakharov-Sagdeev spectrum, the red solid line corresponds to the Kadomtsev-Petviashvili spectrum.



Simulation results: WT

The WT Zakharov-Sagdeev dependence on ϵ. The numerical Kolmogorov-Zakharov constant $C_{KZ} \approx 0.24$. Kinetic isotropic equation gives $C_{KZ} \approx 0.22$.



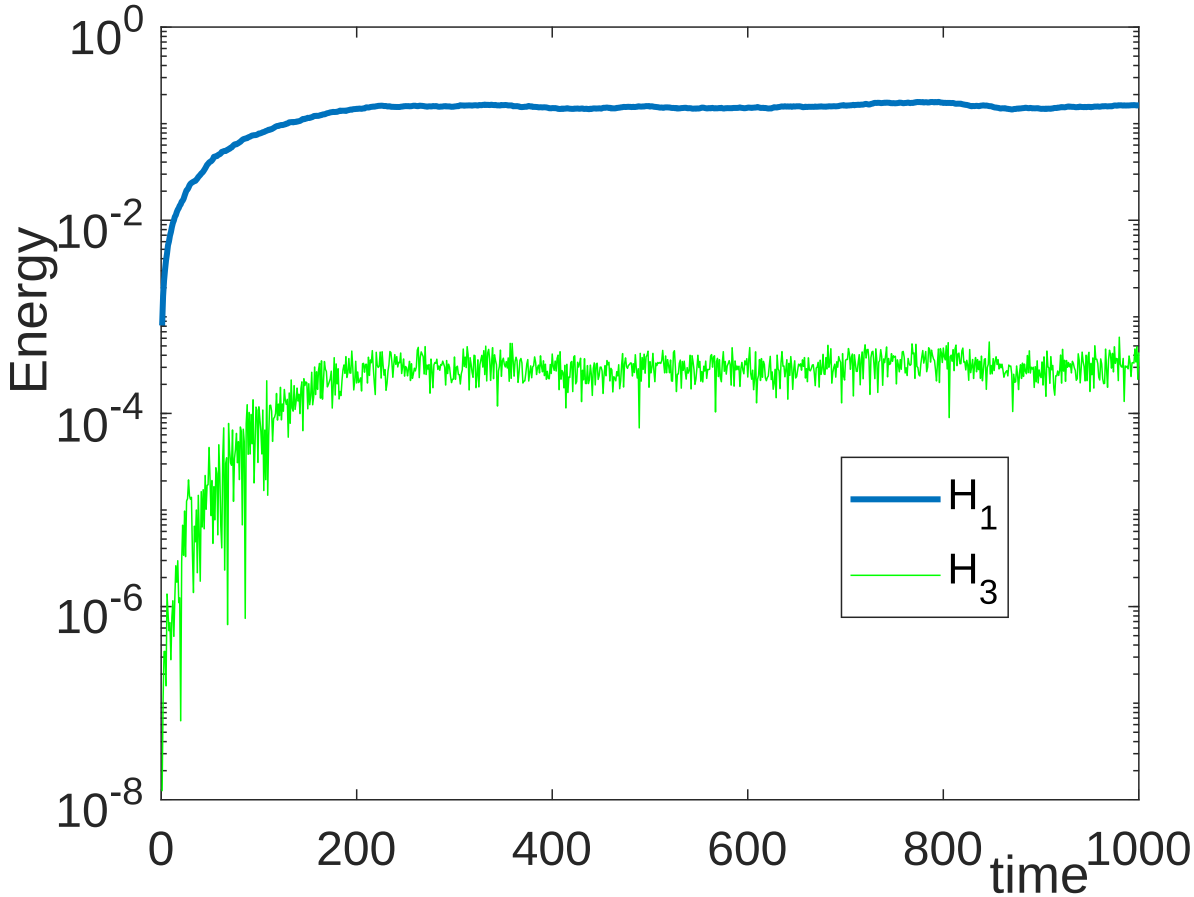
Simulation results: WT

We measured PDF for $|u_x|$ and found that it is is close to a normal Gaussian distribution. The found Skewness $S = \langle u_x^3 \rangle / \langle u_x^2 \rangle^{3/2}$ is near $6.4 \cdot 10^{-3}$, and Kurtosis $K = \langle u_x^4 \rangle / \langle u_x^2 \rangle^2 \approx 3.31$. Deviations of these values from $S = 0$ and $K = 3$ for Gaussian distribution characterize extreme events.

All these results show that we obtain 3D Zakharov-Sagdeev spectrum for weak turbulence regime when wave dispersion is larger than the nonlinearity.

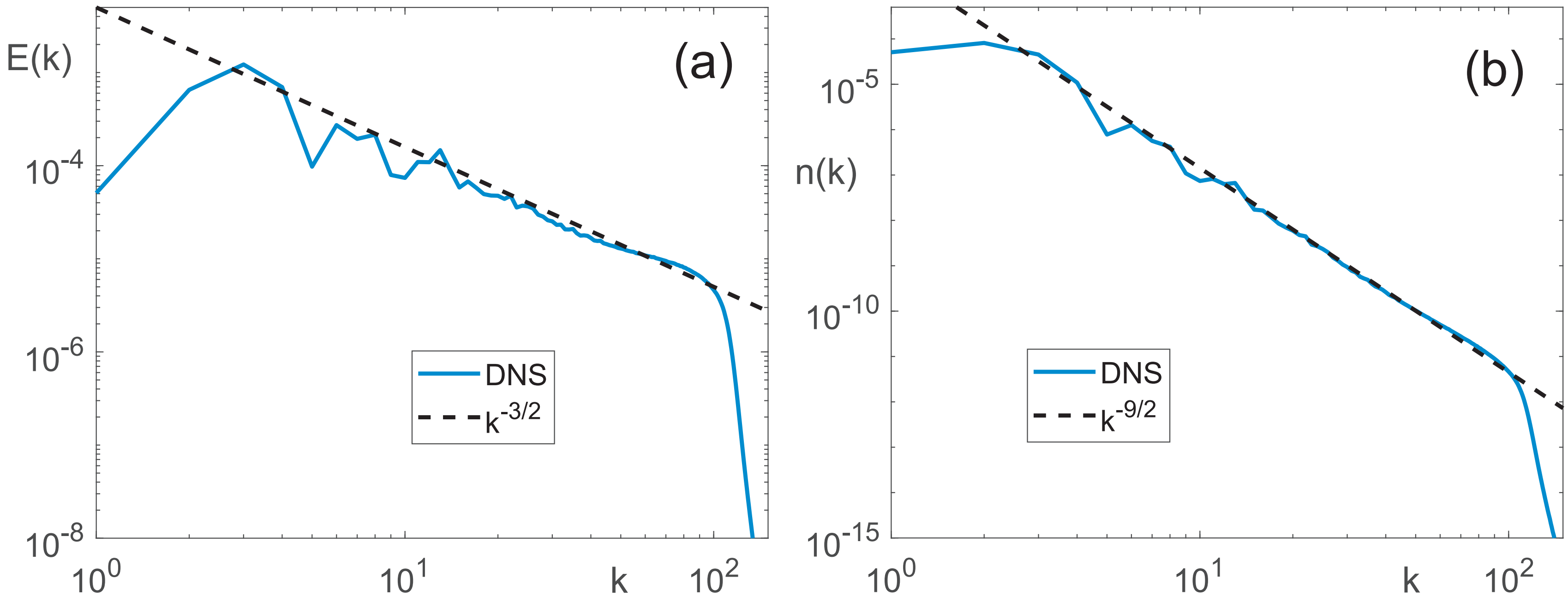
Zakharov-Sagdeev spectrum in dispersionless case $a = 0$

The Hamiltonians H_1 and H_3 in the dispersionless case.
Pumping and damping are the same as in the WT regime.



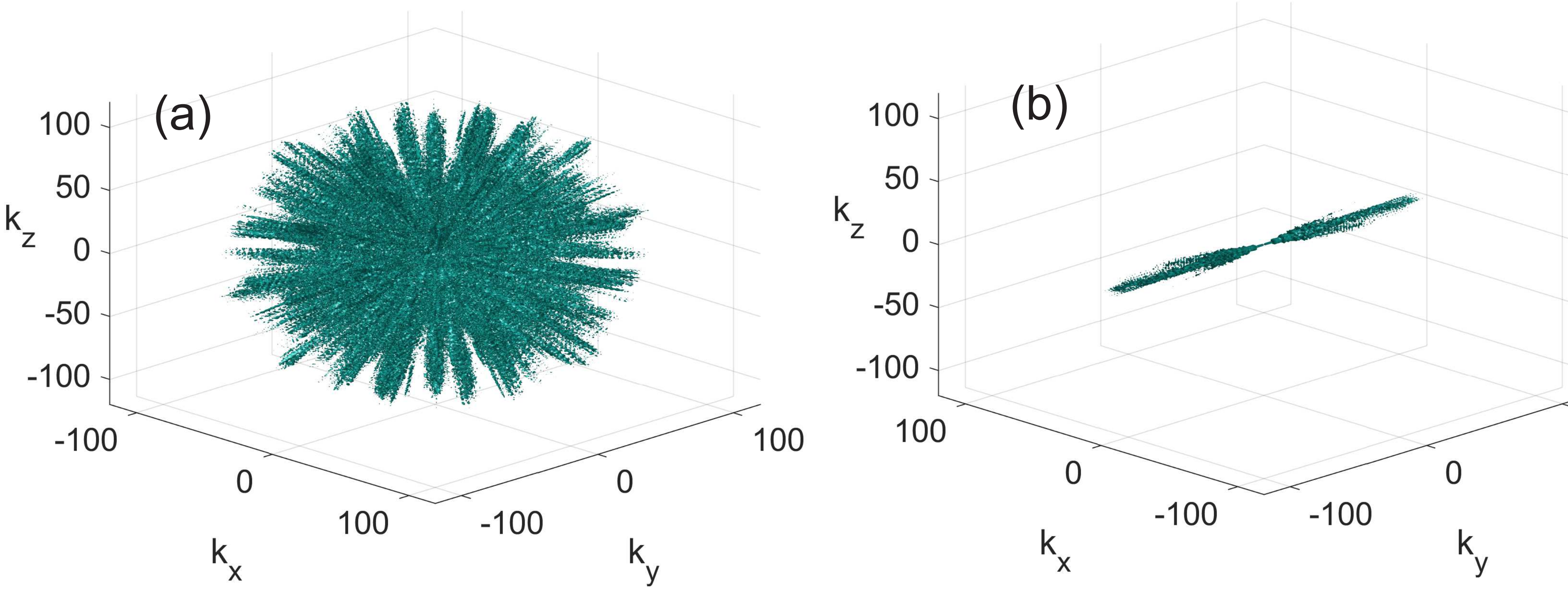
Zakharov-Sagdeev spectrum in dispersionless case $a = 0$

$E(k)$ and $n(k)$ measured in the quasi-stationary state corresponds to the ZS spectrum.



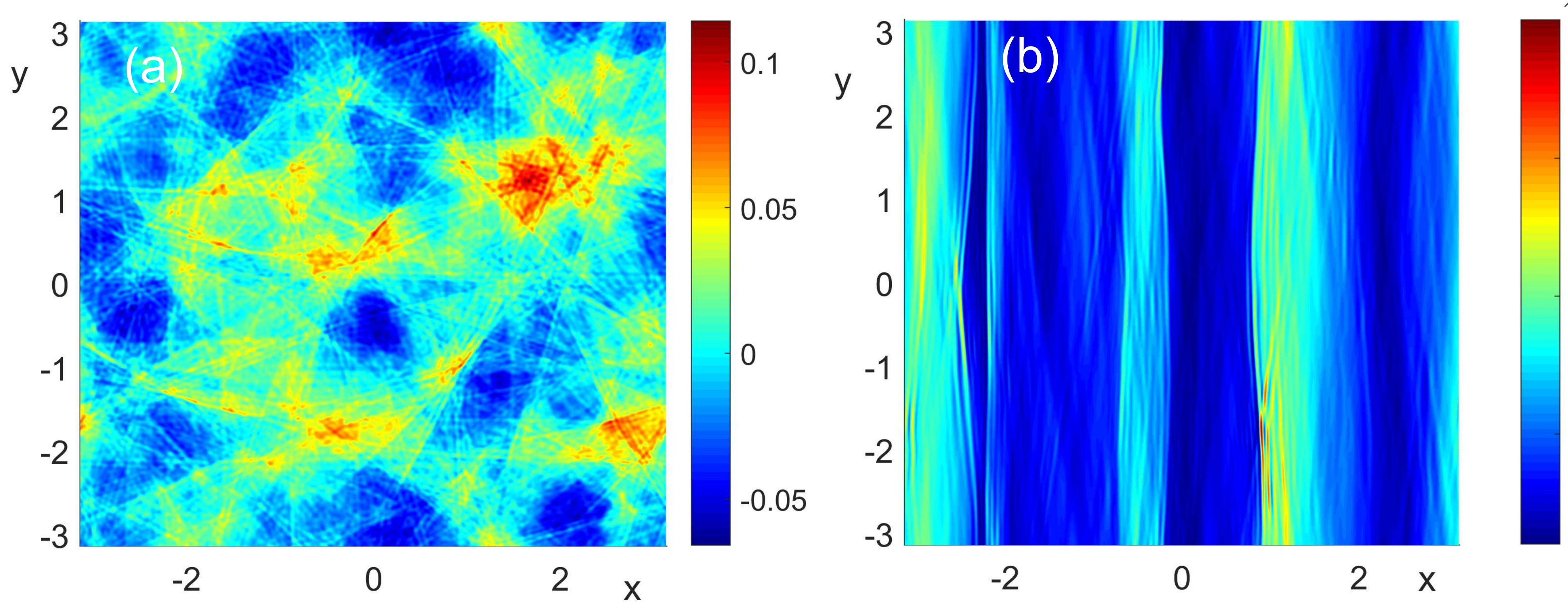
Zakharov-Sagdeev spectrum in dispersionless case $a = 0$

The isosurface of $|u_k|$ (hedgehog) in the quasi-stationary state (left) and the isosurface of one jet (right).



Zakharov-Sagdeev spectrum in dispersionless case $a = 0$

Distribution of $|u(x, y, z = 0)|$ (left) and distribution of $|u_{jet}(x, y, z = 0)|$ (right). For one jet: $H_{dif} \gg H_3$ where $H_{dif} = \int |k_x||a_k|^2(k_\perp^2/k_x^2)d\mathbf{k}$ is the contribution in H due to diffraction of the jet.

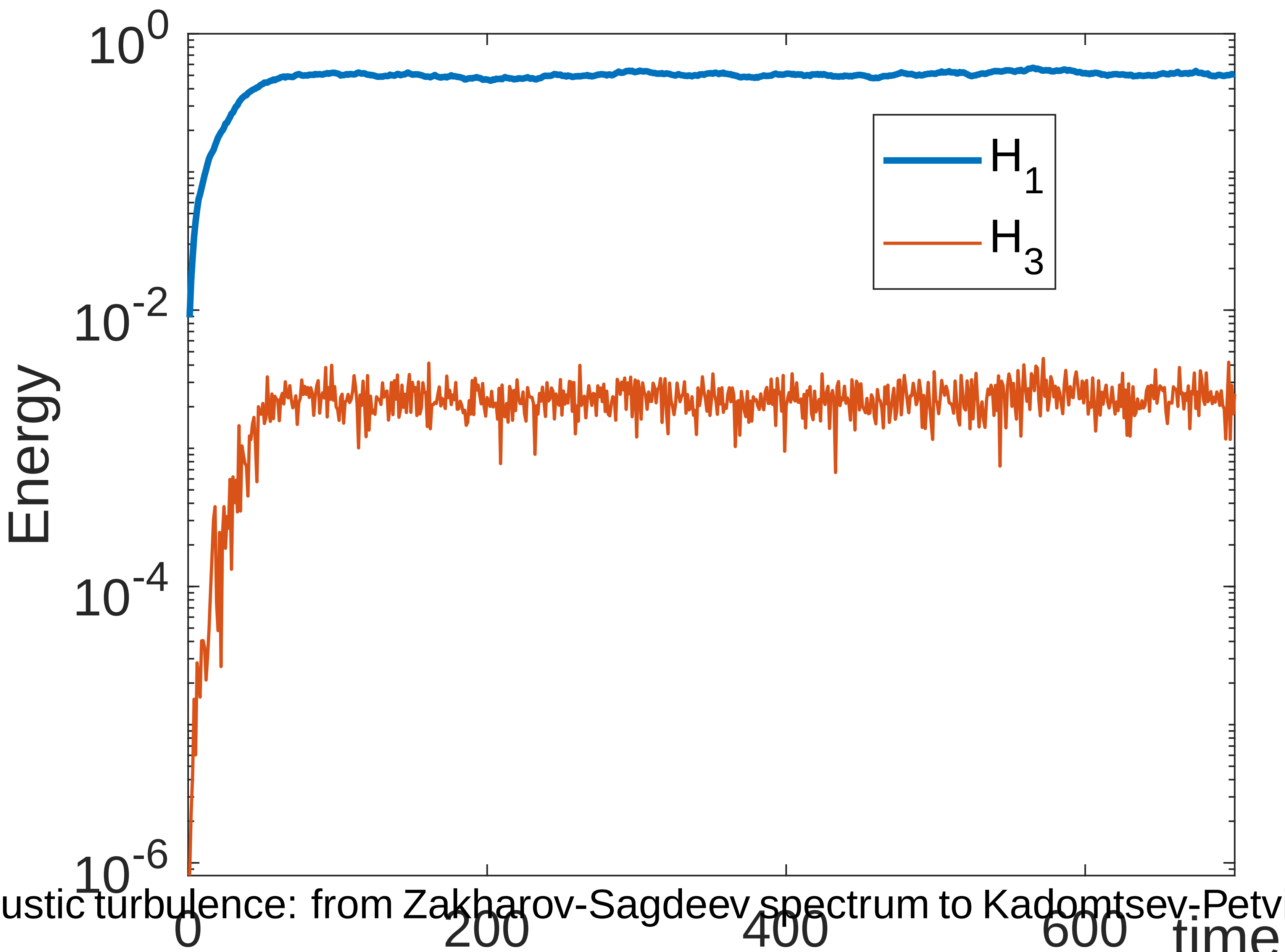


Zakharov-Sagdeev spectrum in dispersionless case $a = 0$

- In this case as for the weak wave dispersion the PDF is close to Gaussian distribution.
  Skewness $S \approx 4.9 \cdot 10^{-2}$ and Kurtosis $K \approx 3.47$.
- Thus, in the dispersionless case we have again Zakharov-Sagdeev spectrum corresponding to weak turbulence regime when diffraction plays the role of wave dispersion.

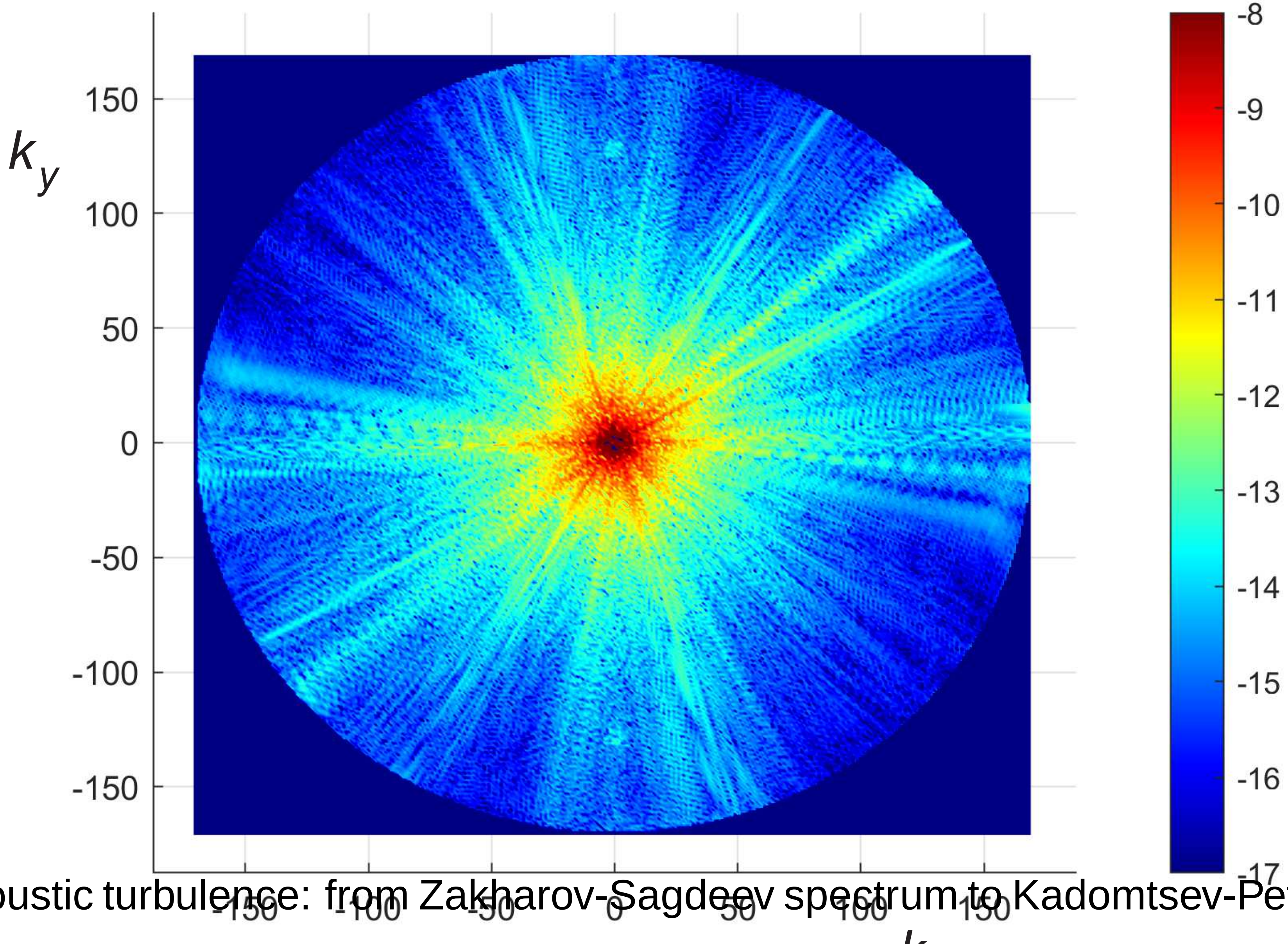
Strong acoustic turbulence

Relations between Hamiltonians



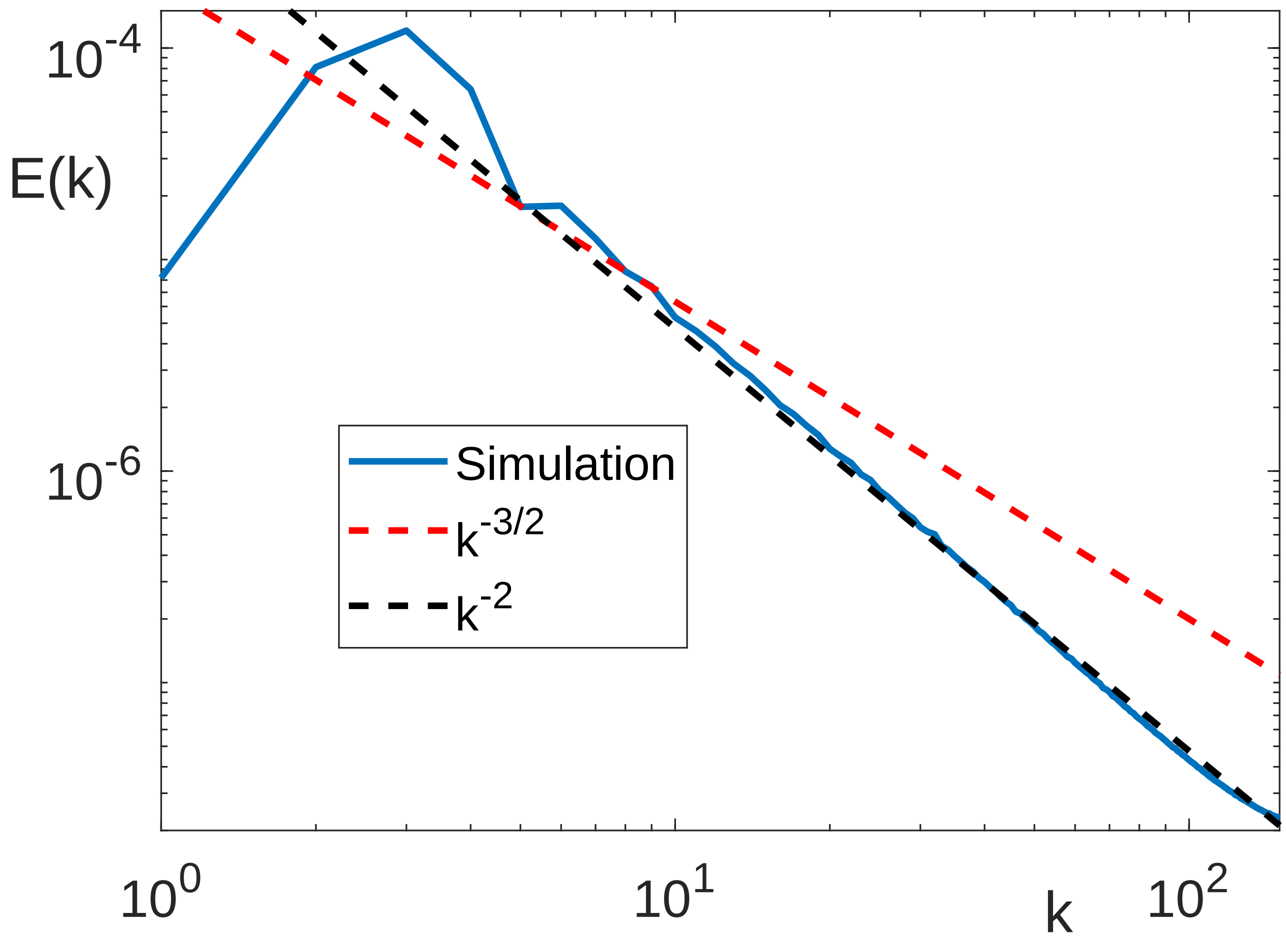
Strong acoustic turbulence

Jets at $k_z = 0$



Strong acoustic turbulence

KP spectrum for $E(k)$.



Strong acoustic turbulence

(a) - 2D distribution $|u(x, y, z = 0)|$ and (b) - 2D distribution $|\nabla u(x, y, z = 0)|$.

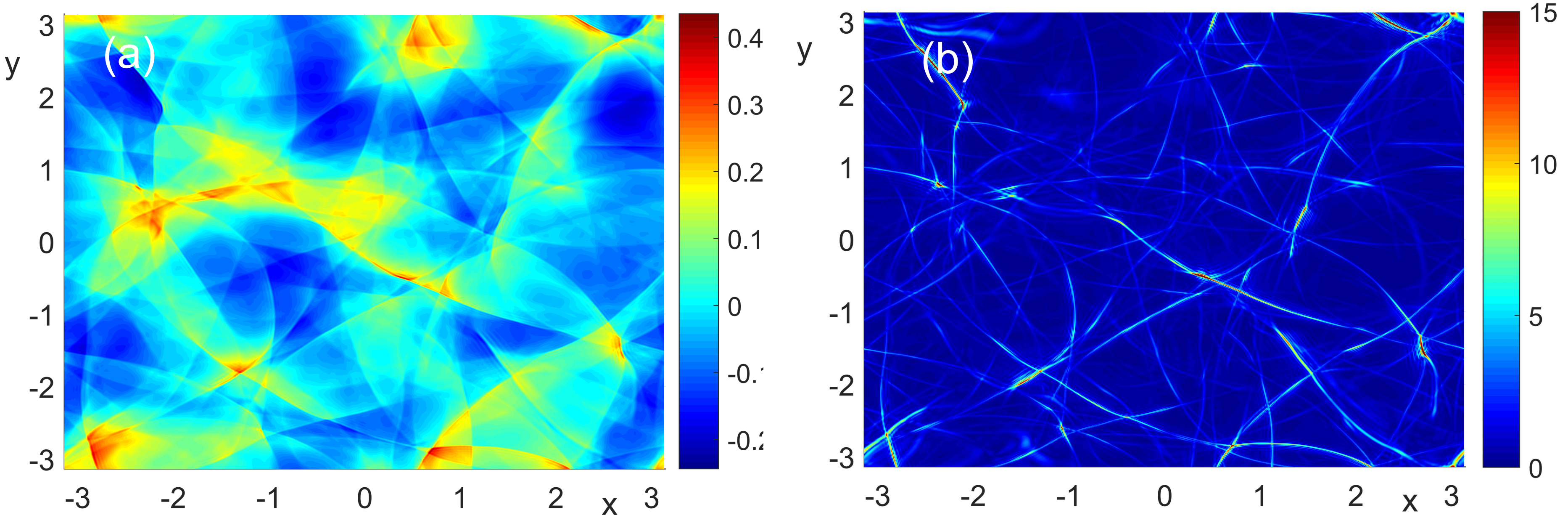
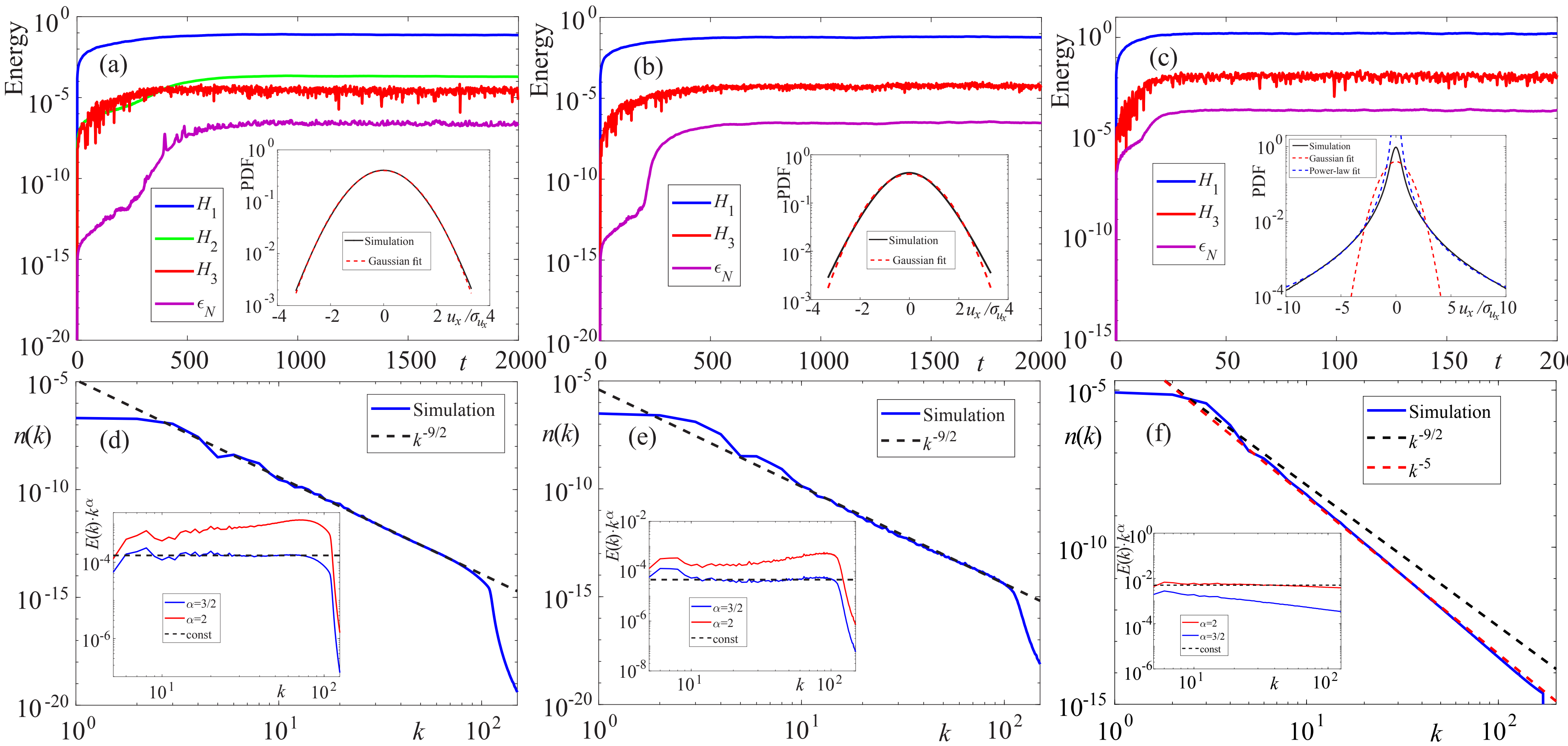


Table for all regimes

Dispersion length $a = 2.5 \cdot 10^{-3}$ (a) and for $a = 0$ (b) and (c).



Conclusion

- We have established that due to the pumping the system of nonlinear interacting weakly dispersive acoustic waves quickly enough passes into the quasi-stationary chaotic state.
- In the quasi-stationary regime, in the long-wavelength region, close to pumping, we observed in the turbulence spectrum the appearance of narrow jets in the form of cones, which expand upon transition to the short-wave region.

Conclusion

- In the region of large k the spectral energy density ε_k tends to an isotropic distribution, for which the dispersion remains weak. In this range of scales, the turbulence spectrum calculated in the stationary state agrees with a high accuracy with the analytical Zakharov-Sagdeev spectrum of weak acoustic turbulence.
- It has been numerically demonstrated that the criteria of weak turbulence are fully satisfied for this spectrum.

Conclusion

- In the dispersionless case we have found the intermediate regime when the Zakharov-Sagdeev spectrum is observed. The energy distribution in the k-space represents a set of jets. Such regime is realized for the same pumping and damping as in the WT situation.
- For this regime we have established that the jet Hamiltonian due to diffraction is of the same order of magnitude as the nonlinear Hamiltonian. This means that diffraction plays a role of wave dispersion in this case.

Conclusion

- With increasing of the pumping amplitude (approximately in 10 times more than in the intermediate regime) we have observed the Kadomtsev-Petviashvili spectrum $E(k) \propto k^{-2}$. The turbulence state in this case is a random set of shocks.
- The results obtained are the first reliable observations of the spectrum of both weak and strong turbulence of acoustic waves in media with positive dispersion and dispersionless media in direct three-dimensional numerical simulations.

THANKS FOR YOUR ATTENTION