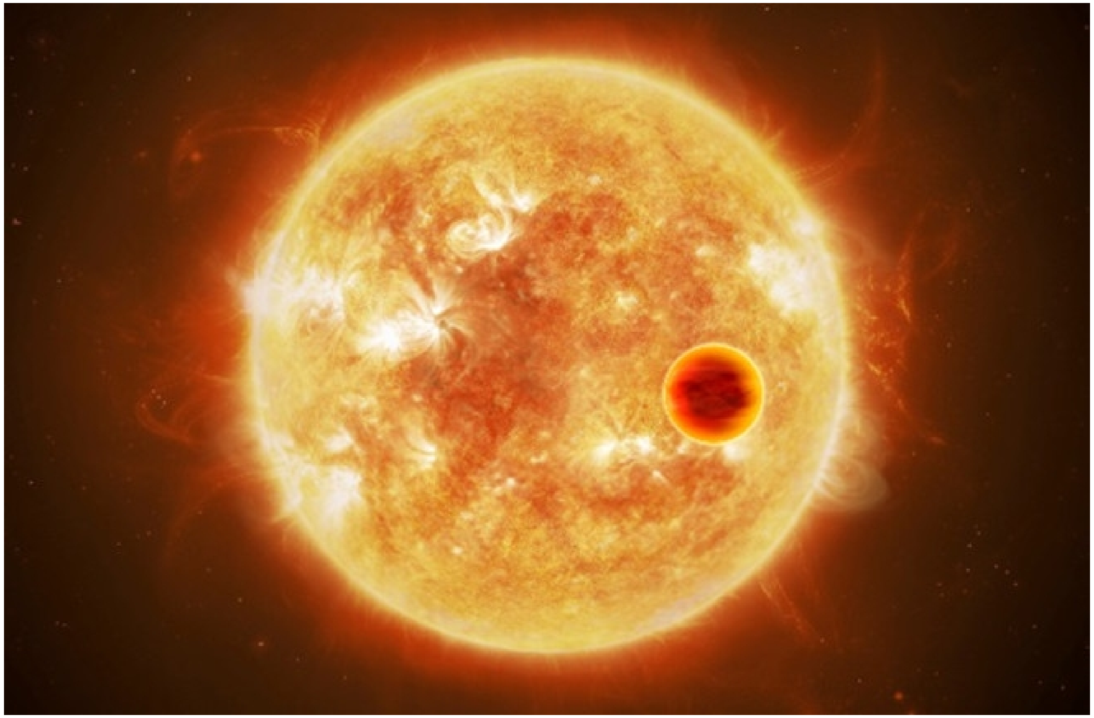
Tidal interactions and waves in stars

Pavel Ivanov, Lebedev Physical Institute



Dynamic and quasi-static tides

- In principle, there are two contributions due to tidal interactions, which can be easily separated:
- 1) quasi-static tides. The energy and angular
- momentum are transferred between the orbit and the star on a time characteristic scale determined by orbital motion by viscous forces. The main uncertainty - a value of “tidal Q” defined as the ratio of energy stored in tidal bulge to the amount of energy dissipated per forcing period.

- 2) Dynamic tides. During the periastron passage oscillations are excited at frequencies corresponding to a star's normal
- mode frequencies due to resonant interactions. These are: fundamental modes with frequencies $\omega \sim \Omega_* = \sqrt{GM/R_*^3}$, inertial, and gravity waves with $\omega << \Omega_*$ in case of stably stratified stars. In certrain regimes the outcome of dynamic tides does not depend on viscosity of stellar gas. However, to justify these regimes some non-linear mechanisms of dissipation of tidal energy must be invoked.
-

A simple model problem

Let us consider the standard linear forced oscillator equation

$$\ddot{x} + \gamma \dot{x} + \omega_*^2 x = f(t),$$

where we assume that the dissipation rate γ is much smaller that the eigen frequency ω_*. We also assume that $\omega_f \equiv \frac{|\dot{f}|}{|f|} \ll \omega_*$, but, $\omega_f \gg \gamma$.

In general there are two contributions to the solution corresponding to 'quasi-static and dynamic tides', respectively. The former describes a part of the solution evolving with characteristic time $\sim \omega_f^{-1}$, while the latter describes the part evolving on the timescale ω_*^{-1}.

In order to find the first part we look for the solution in the form $x = \frac{1}{\omega_*^2} f + x_1$, and easily find that $x_1 \approx -\frac{\gamma \dot{f}}{\omega_*^4}$. We have, accordingly,

$$x(t) \approx \frac{1}{\omega_*^2} f - \frac{\gamma \dot{f}}{\omega_*^4} \approx \frac{1}{\omega_*^2} f(t - \frac{\gamma}{\omega_*^2}).$$

The last equality shows that we have successfully modelled the well-known 'tidal lag' associated with quasi-static tides.

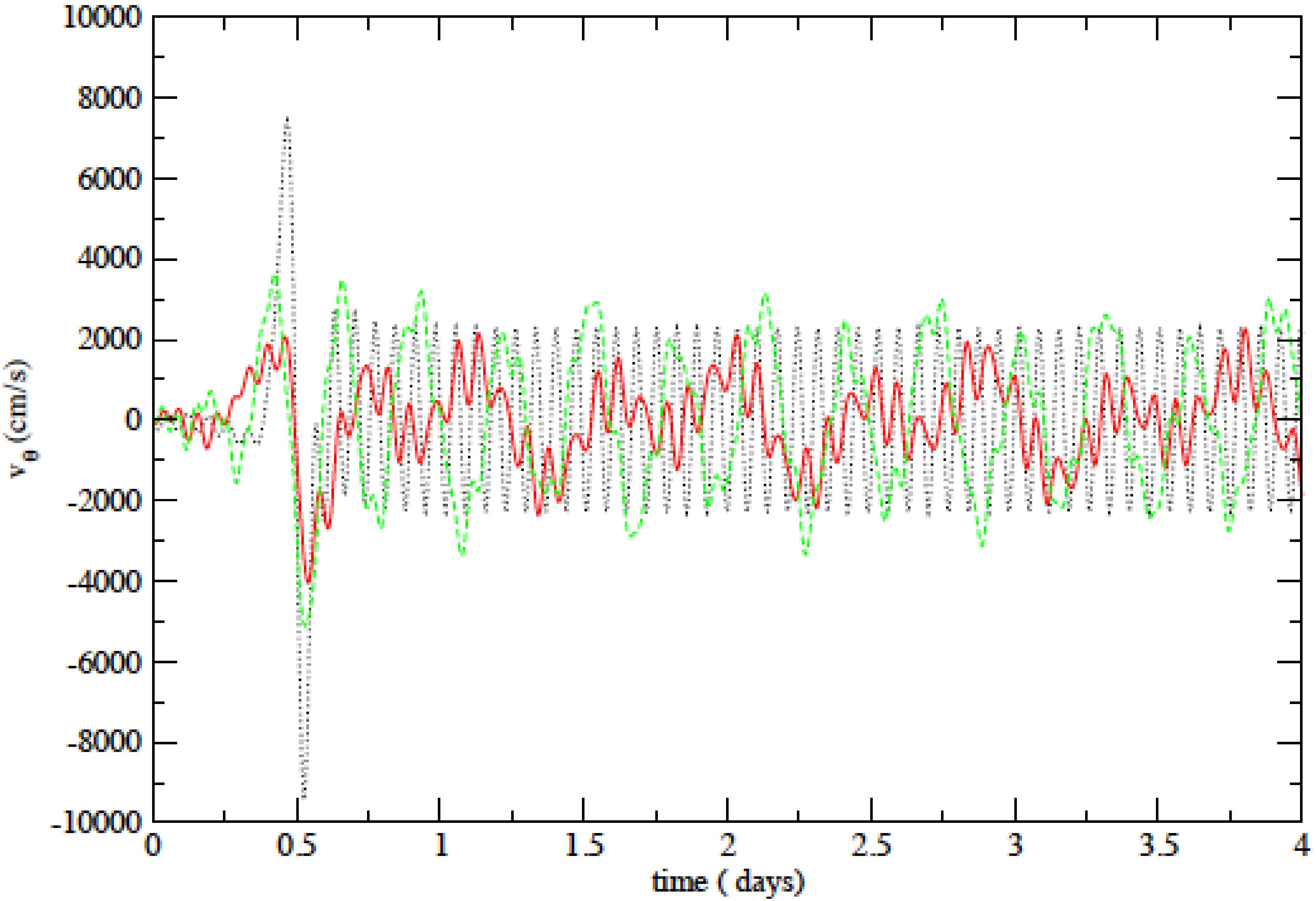
To find the second part we do Fourier transform of the forcing term $f = \frac{1}{2\pi} \int d\omega f(\omega) e^{i\omega t}$ and obtain the formal solution

$$x(t) = \frac{1}{2\pi} \int d\omega \frac{f(\omega) e^{i\omega t}}{(\omega_*^2 - \omega^2 + i\omega\gamma)}.$$

When considering times $\gg \omega_f^{-1}$ only the poles in the integral matter and we approximately have

$$x(t) \approx -2\pi i \sum_{\pm} f(\omega_{pm}) e^{i\omega_{\pm} t},$$

where $\omega_{\pm} = \pm\omega_* + i\gamma/2$. Clearly, we have modelled excitation of free oscillations by the forcing term. The same effect takes place when dynamic tides are considered.



10000
8000
6000
4000
2000
0
-2000
-4000
-6000
-8000
-10000
v_θ (cm/s)
0
0.5
1
1.5
2
2.5
3
3.5
4
time (days)

Forced oscillations of a star

When studying linear oscillations of a star it is convenient to make the Fourier transform $\xi(t) \rightarrow \xi(\omega)$.

(Note that when the forcing is nearly periodic, one should use Fouries series!)

Let us consider a uniformly rotating star. In the rotating frame our problem can be formulated as

$$\omega^2\xi - \omega\mathbf{B}\xi - \mathbf{C}\xi = S(\omega),$$

where $\mathbf{B}\xi$ and $\mathbf{C}$ are self-adjoint. Moreover, in certain cases it is possible to assume that $\mathbf{C}$ is non negative definite, i. e. $< \xi|\mathbf{C}\psi > \geq 0$ for any ξ and ψ.

Let me consider square root of $\mathbf{C}$, $\mathbf{C}^{1/2}$ defined by condition: $\mathbf{C} = \mathbf{C}^{1/2}\mathbf{C}^{1/2}$.

Also, let me introduce a six dimensional vector $\vec{Z}$ with components:

$$Z_1 = \omega\xi, \quad Z_2 = \mathbf{C}^{1/2}\xi$$

Then, it is easy to see that the problem can be reformulated in a way associated with a new self adjoint operator $\mathcal{H}$

$$\omega\vec{Z} = \mathcal{H}\vec{Z} + \vec{S},$$

where $\mathcal{H}$ has a matrix structure

$$\mathcal{H} = \begin{pmatrix} \mathbf{B} & \mathbf{C}^{1/2} \\ \mathbf{C}^{1/2} & 0 \end{pmatrix},$$

and the source six dimensional vector $\vec{S}$ has the components

$$S_1 = S(\omega), \quad S_2 = 0$$

The solution is expressed in terms of eigen vectors of $\mathcal{H}$:

$$\vec{Z} = \sum_k \alpha_k \vec{Z}^k,$$

where $\vec{Z}_k$ satisfy

$$\omega_k \vec{Z}^k = \mathcal{H}\vec{Z}^k.$$

$\vec{Z}^k$ are orthogonal in the sense of the inner product

$$< \vec{Z}^k | \vec{Z}^l >= \omega_k \omega_l (\xi_k | \xi_l) + (\xi_k | \mathbf{C} \xi_l).$$

$$\alpha_k = \frac{< \vec{Z}_k | \vec{S} >}{N_k(\sigma + i\sigma_\nu - \sigma_k)},$$

where $N_k =< \vec{Z}_k | \vec{Z}_k >$ is the norm.

The displacement vector ξ is obtained from its Fourier transform $\tilde{\xi}_m$ by integration over σ and summation over m

$$\xi = \sum_{m,k} \int \mathrm{d}\sigma \left[\frac{\sigma_k S_k}{N_k(\sigma + \mathrm{i}\sigma_\nu - \sigma_k)} \xi_k \mathrm{e}^{-\mathrm{i}\sigma t} \mathrm{e}^{\mathrm{i}m\phi} + c.c. \right]. \tag{27}$$

This expression can be simplified in the limit $t \to \infty$ at which only the poles in the expression in the braces contribute significantly to the integral over σ. In this limit we may, therefore, write

$$\xi = 2\pi\mathrm{i} \sum_{m,k} \left(\frac{\sigma_k S_k}{N_k} \mathrm{e}^{-\sigma_\nu t - \mathrm{i}\sigma_k t} \mathrm{e}^{\mathrm{i}m\phi} \xi_k + c.c. \right). \tag{28}$$

The same formalism is working in case when the source term is (quasi) periodic. The only difference is that we should work with Fourier series instead of Fourier transform. The basic relations in this case are very simple in the case of a regular dense eigen spectrum (e.g. internal gravity waves) and the so-called 'moderately large viscosity', when the ratio of an effective dissipation rate to the difference between two neighbouring eigen frequencies, κ, is larger than one.

$$\kappa = \left| \frac{\omega_{\nu,kj_0}}{\mathrm{d}\omega_{j_0}/\mathrm{d}j_0} \right| > 1$$

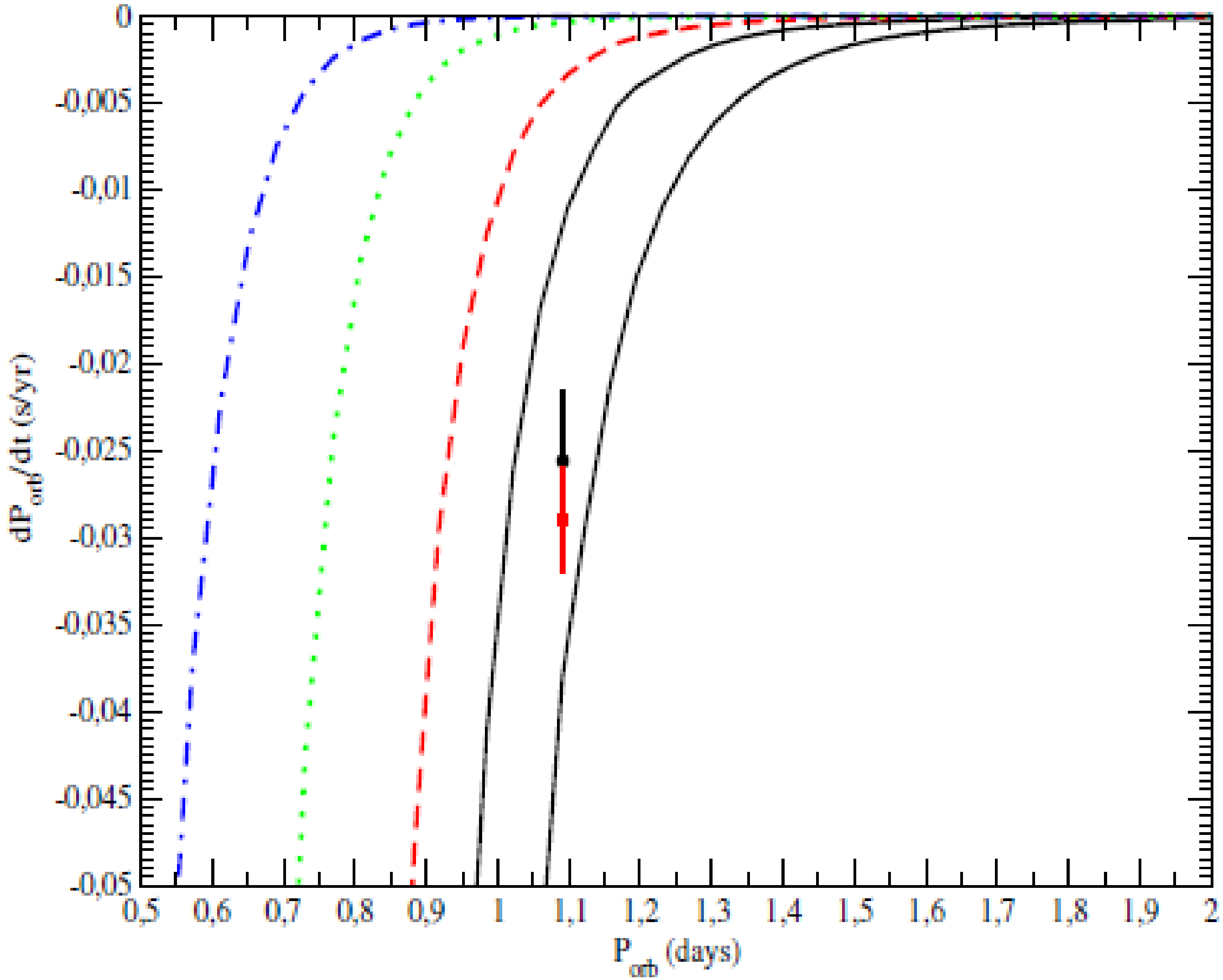
In this case the energy and angular momentum transfer rates from the orbit to the modes, and then to a star does not depend on viscosity

$$\dot{E}_c = -4\pi^2 \sum_{m,k} \frac{\omega_{m,k}^2}{|\mathrm{d}\omega_{j_0}/\mathrm{d}j_0|} \frac{|S_{j_0,k}|^2}{N_{j_0}}$$

$$\dot{E}_c = -\pi \sum_{m,k} \frac{|A_{m,k}\hat{Q}_{j_0}|^2}{|\mathrm{d}\omega_{j_0}/\mathrm{d}j_0|},$$

$$\dot{L}_c = -\pi \sum_{m,k} \frac{m}{\omega_{m,k}} \frac{|A_{m,k}\hat{Q}_{j_0}|^2}{|\mathrm{d}\omega_{j_0}/\mathrm{d}j_0|} \quad \text{and}$$

$$\dot{E}_I = -\pi \sum_{m,k} \left(1 + \frac{m\Omega}{\omega_{m,k}}\right) \frac{|A_{m,k}\hat{Q}_{j_0}|^2}{|\mathrm{d}\omega_{j_0}/\mathrm{d}j_0|}.$$



An example of application of this formalism to the system WASP-12b containing a Hot Jupiter (Ivanov, Papaloizou, Chernov, 2017).

The central issue of the whole theory is, however, how to provide $\kappa > 1$? One can see, that, the linear mechanism such as e.g. convective/radiative viscosity cannot do the job for stars on the main sequence. Therefore, it's natural to invoke non-linear mechanisms.

In what follows I am going to consider only g-modes in MS stars. The central quantity, which governs these waves is the Brunt–Väisälä frequency N defined according to the rule

$$N^2 = g_0 \left(\frac{1}{\Gamma_{1,0}} \frac{\mathrm{d} \ln p_0}{\mathrm{d}r} - \frac{\mathrm{d} \ln \rho_0}{\mathrm{d}r} \right)$$

It may be shown that $N^2 = A ds/dr$, where s is specific entropy and $A > 0$.

In particular, eigen frequencies of g-modes in WKBJ approximation obey

$$\omega = \frac{\sqrt{\Lambda} I}{\pi(n + (1+l)/2 + q/(4(2+q))) + \Delta\phi(\omega_0)}$$

where n is the mode's order, l and Λ are 2 and 6 for a non-rotating star,

$$I = \int_0^{r_c} \frac{\mathrm{d}r}{r} N$$

and the other terms provide some corrections.

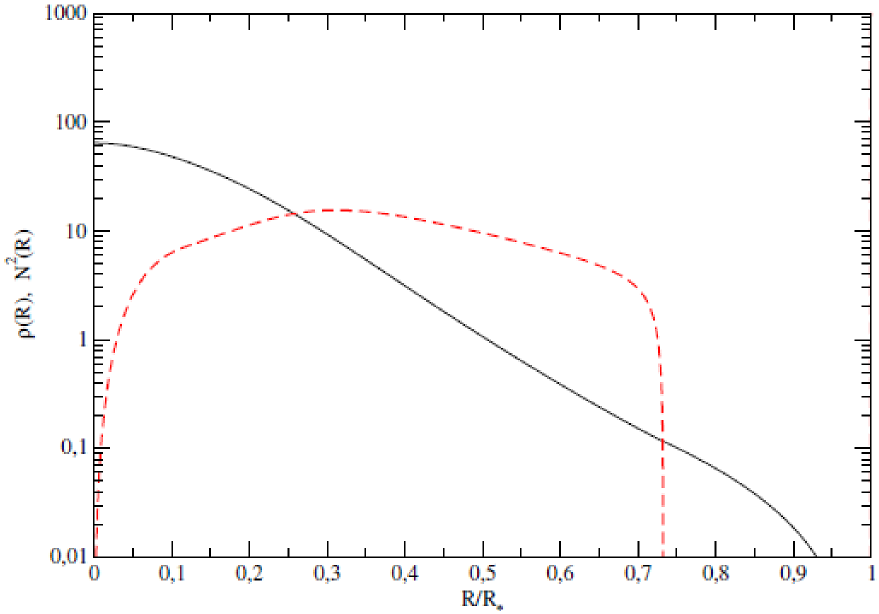


Figure 6. Radial profiles of stellar density ρ (black solid line) and square of the Brunt-Väisälä frequency N^2 (red dashed line), both in the natural units defined in § 2, for model 1a.

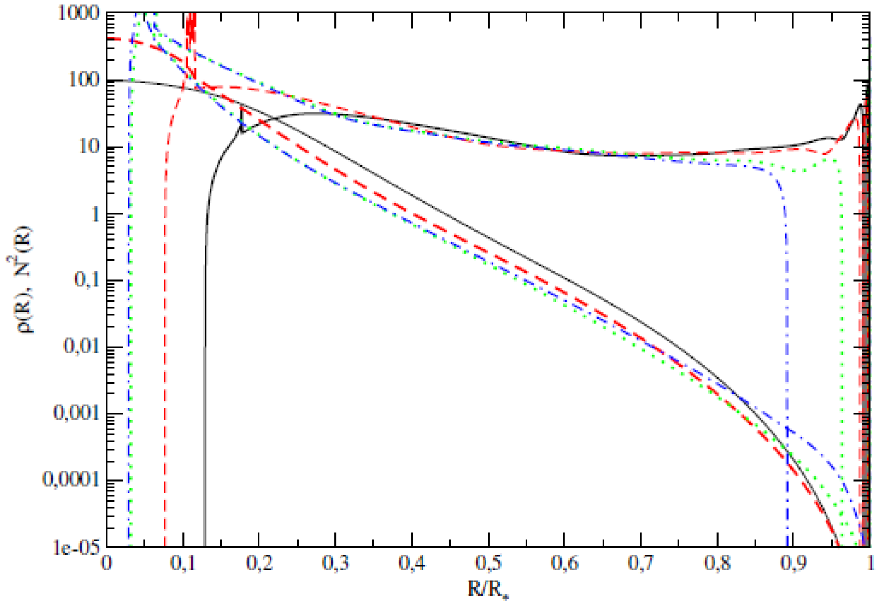


Figure 8. Same as Fig. 7, but for the models 2a, 2b, 2c and 2d shown as black solid, red dashed, green dotted and blue dot-dashed lines, respectively.

Typical distributions of BV frequency and density (in natural units): left panel - a solar model, right panel – models of stars with masses equal to 2 solar masses.

Let us first consider a star of solar type (with radiative interior). In such stars g-modes can propagate down to the centre, where they can 'break'. The condition for wave Breaking (which is typically justified by some numerical arguments) is that the absolute value of the gradient of Eulerian perturbation of specific entropy

$\partial_r s'$ is larger than its background value ds/dr and, accordinlgy, perturbation of BV frequency is formally larger than its background value.

Taking into accout that Largangian perturation of entropy Δs is zero for adiabatic perturbations, and that $\Delta s = s' + \frac{ds}{dr}\xi^r$

this condition can be translated to the condition

$$\left| \frac{\partial}{\partial r}\xi^r + \frac{\frac{d^2 s}{dr^2}}{\frac{ds}{dr}}\xi^r \right| > 1$$

and the expression on l.h.s. can be evaluated when $\dot{E}_I$ is known.

Assuming that $N = A_{\rm centre} r$, we obtain the condition

$$\frac{q}{1+q} > C_{crit},$$

where

$$C_{\rm crit,centre} = \frac{320}{9\sqrt{2}} \frac{\rho_{\rm centre}^{1/2}}{I^{1/2} A_{\rm centre}^{5/2}} \frac{1}{\hat{Q}} \Omega_{\rm orb}^3$$

and $\hat{Q}$ are the so-called overlap integrals. They are functions of the orbital frequency and a stellar model. q is mass ratio of the stars.

But, more massive stars have convective cores, the wave breaking mechanism doesn't work efficiently, and the coefficient C_{centre} turns to be too large.

What to do? One way out of this trouble is to consider effects near the interface between outer convective envelope and the intermediate radiative zone (Ivanov, Chernov, Barker, 2022). For that we consider a model probem near the interface radius, r_c.

We assume the vertical velocity to be zero at some distances $-z_c$ and z_r in the neutral and radiative zones, respectively. This formulates an eigen problem, which is solved iteratively, for harmonics of first and second order, respectively.

- We also assume the Boussinesq approximation, the planar geometry and that the square of Brunt-Vaisala frequency decreases linearly with the distance to the interface radius.
- $$\dot{\mathrm{U}}+(\mathrm{U}\cdot\nabla)\mathrm{U}=-\nabla P+b\mathrm{e}_z, \quad \nabla\cdot\mathrm{U}=0, \quad \dot{b}+(\mathrm{U}\cdot\nabla)b=-zU^z. \qquad (27)$$

The set of equations is solved iteratively, in the linear approximation, and in the next order, where the linear solution is used as source. We then check, under which condition the amplitude of the second order perturbations is the same is an amplitude of the first order perturbations. It is assumed that when this happens, a non-linear instability sets in (but, it has not been checked!!!).

The second order solution is looked for as a decomposition over the free pulsations, in full analogy with the tidal problem, and it turns out that there is a resonance for a mode, which has its eigen frequency twice a value of the primary mode. However, this resonance is not exact, different factors (albeit small ones) limit the amplitude of the secondary mode. Taking into account these factor allows us to formulate another criterion of a possible strongly non-linear behaviour of the system in the same manner as the previous one. Namely, when the mass rato q is such that $q/(1+q) > C_{crit,c}$, we assume that the system becomes non-linear and tides are possibly damped.

$$\ddot{a}_j + \lambda_j a_j = -\frac{\lambda_j}{k^2}\frac{S_j}{N_j}, \tag{52}$$

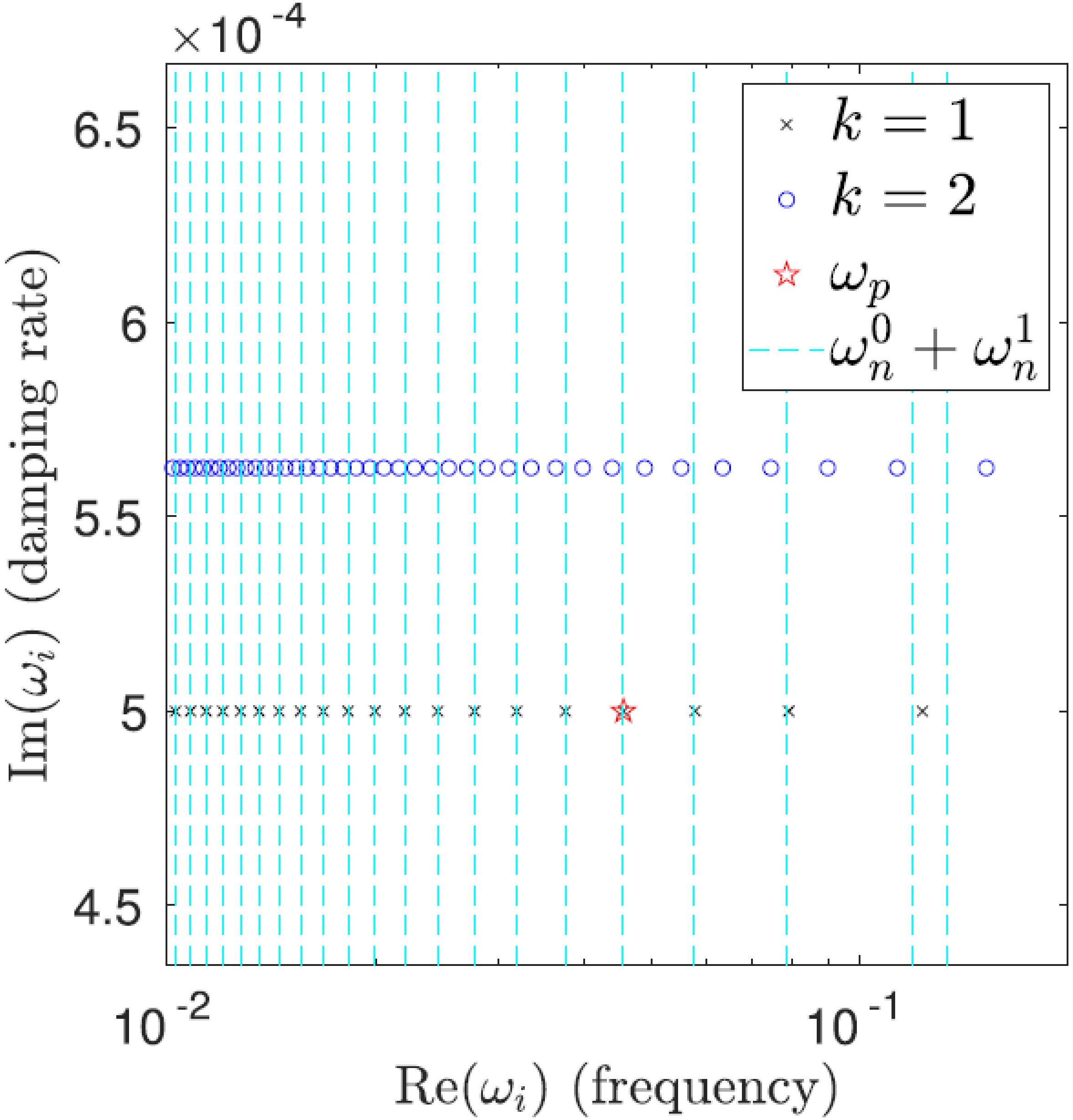
where

$$S_j = \int \phi_j^* S dx dz, \quad N_j = N_{j,j} = \int z \phi_j^* \phi_j dx dz. \tag{53}$$

Remembering that $S \propto e^{2i\omega_p t}$, solutions to (52) are

$$a_j = \frac{\omega_j^2}{k_s^2(4\omega_p^2 - \omega_j^2)}\frac{S_j}{N_j}e^{2i\omega_p t}, \tag{54}$$

where we remember that $\lambda_j = \omega_j^2$.



×10^-4
6.5
6
5.5
5
4.5
Im(ωi) (damping rate)
10^-2
10^-1
Re(ωi) (frequency)
k = 1
k = 2
ωp
ω0n + ω1n

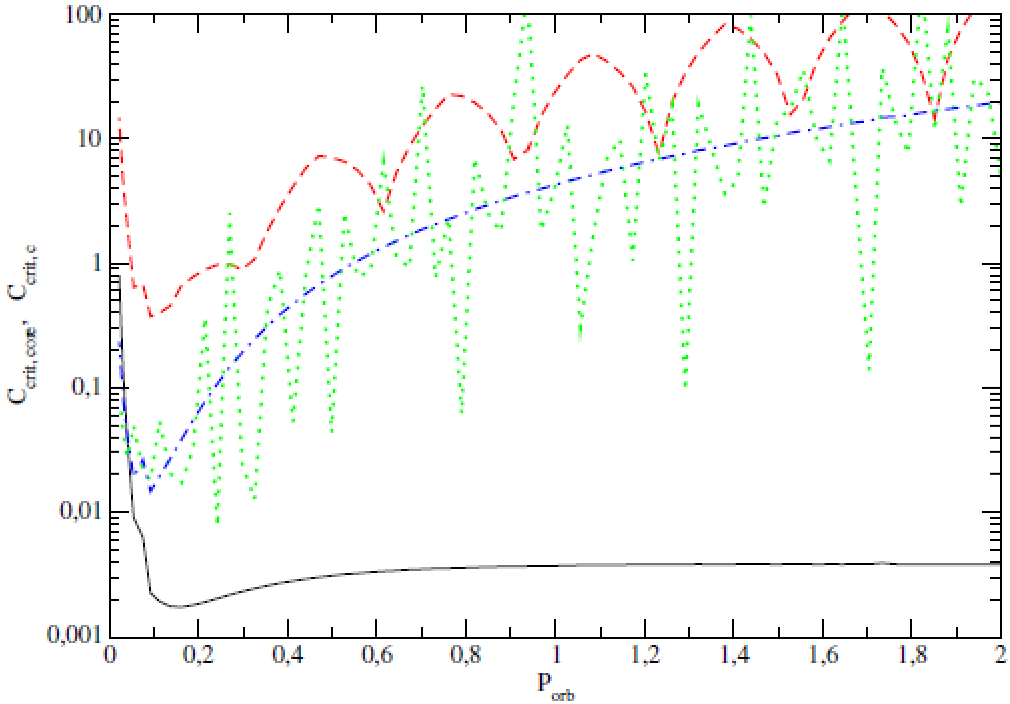
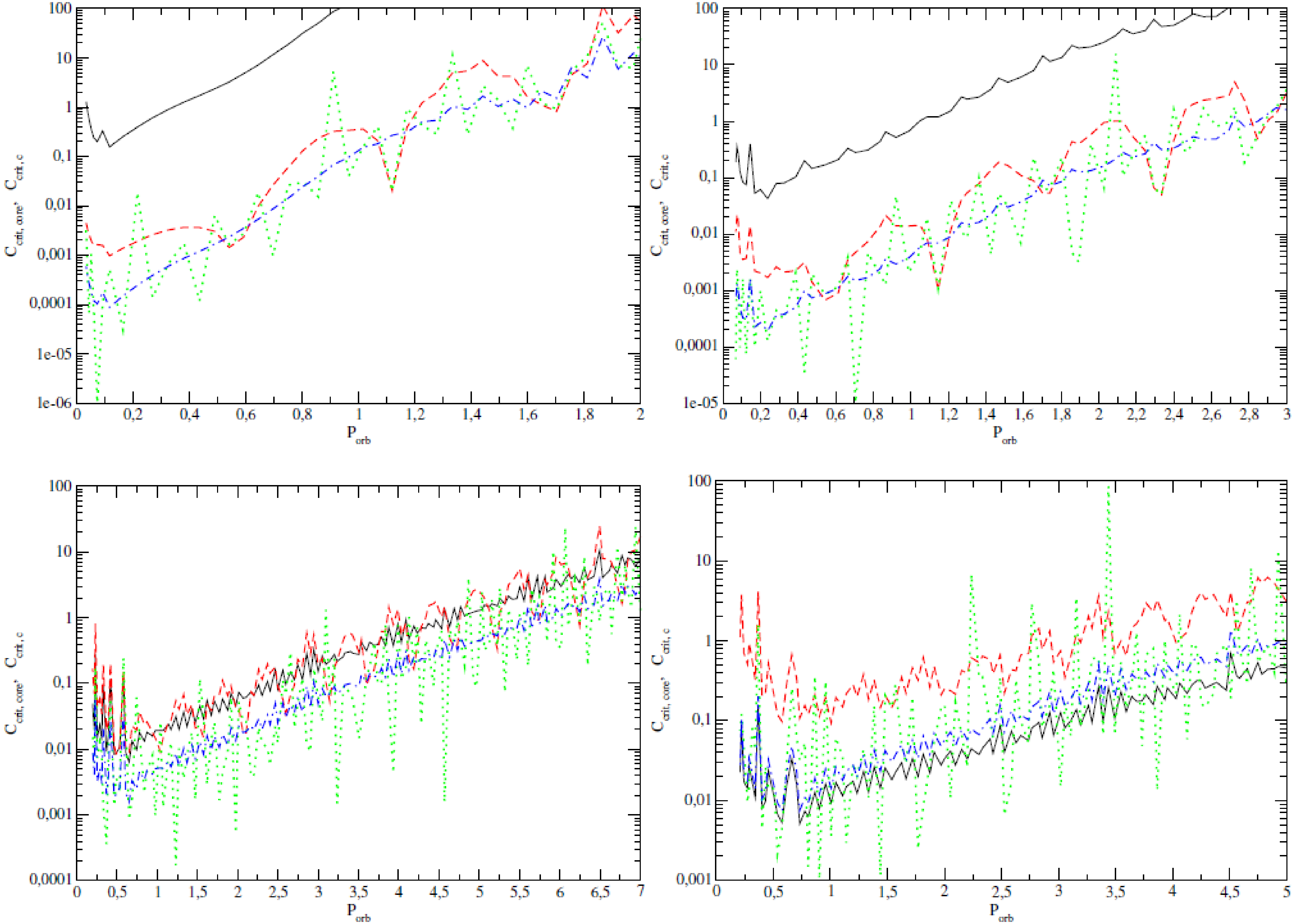
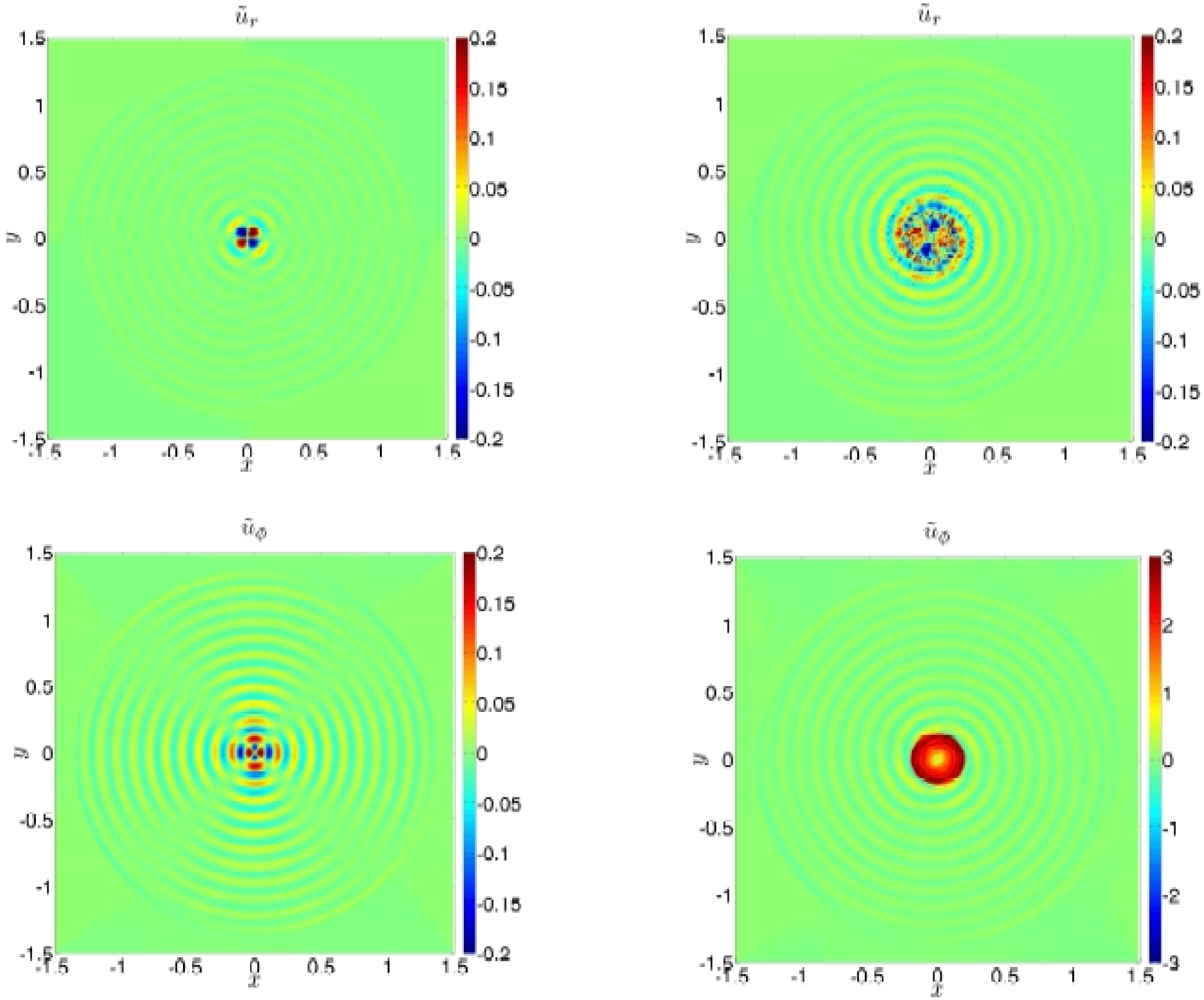


Figure 9. Various criteria for transition to nonlinearity C_{crit} are shown for a solar mass model as a function of orbital period in days. The black solid curve corresponds to C_{centre} using the 'standard' expression (22) for wave breaking in radiative cores, the red dashed one corresponds to $C_{crit,c}$ in (71), the green dotted and blue dot-dashed curves are given by $C^{dense}_{crit,c}$ in eq. (75), where in the former we set $\kappa = 0$ and in the latter $\kappa = 1$.



100
10
1
0,1
0,01
0,001
0,0001
1e-05
1e-06
0
0,2
0,4
0,6
0,8
1
1,2
1,4
1,6
1,8
2
2,2
2,4
2,6
2,8
3
P_orb
C_crit, core, C_crit, c
0,5
1,5
2,5
3,5
4
4,5
5
5,5
6
6,5
7

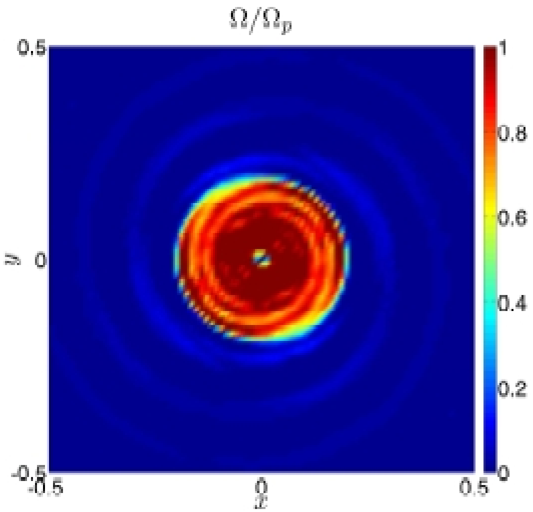
Numerical experiments – as far as I am aware of they all have been done for models corresponding to solar-like stars, wave breaking in the centre and in Boussinesq approximation (sometimes, in cylindrical geometry as well).

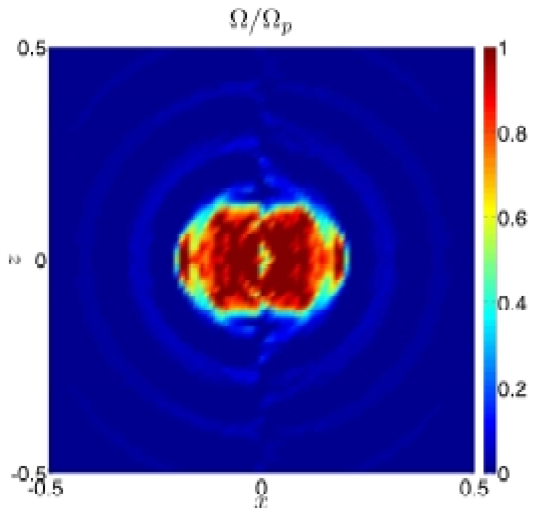


3D simulations (Barker, 2011). The left panels – forcing amplitude is weak and the waves evolve in the linear regime, right panels – forcing is strong enough for wave breaking to occur.

The formation of a critical layer in case of wave breaking

Not only energy, but also angular momentum is transferred by waves. When they break a rotating mean flow is formed. When the angular frequency of this flow is larger when the wave pattern speed ω/m, waves are absorbed near the boundary of the flow.





This is because close to the layer wavelength tends to zero as described in 1D by Taylor-Goldstein equation.

$$\hat{\psi}'' + \{\frac{N^2(z)}{(U(z)-c)^2} - \frac{U''(z)}{U(z)-c} - k^2\}\hat{\psi} = 0.$$

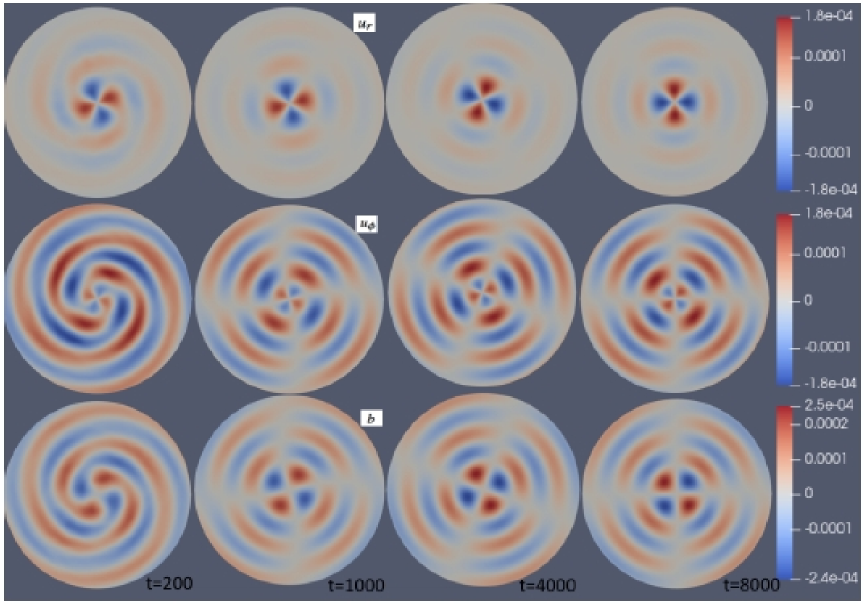


Figure 8. u_r and u_ϕ (radial and azimuthal velocities) and b (buoyancy perturbation) for a simulation with low-amplitude forcing with frequency ω
The diffusion coefficients are $\nu = 10^{-6}$ and $\kappa = 5 \times 10^{-6}$.

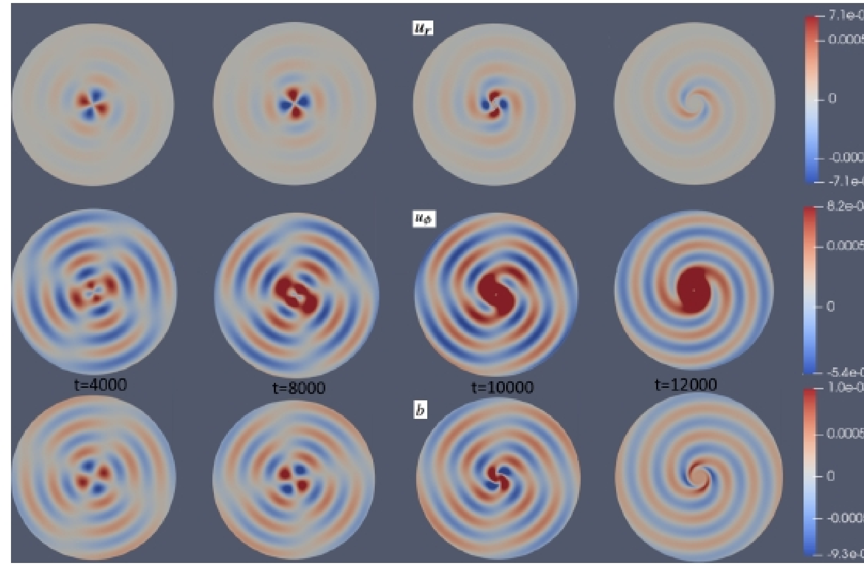


Figure 11. u_r, u_ϕ (radial and azimuthal velocities) and b (buoyancy) for a simulation with intermediate-amplitude forcing $U = 3 \times 10^{-5}$ with frequency $\omega = 0.100$ (corresponding to the black curve in Fig. 9). The diffusion coefficients are $\nu = 10^{-6}$ and $\kappa = 5 \times 10^{-6}$.

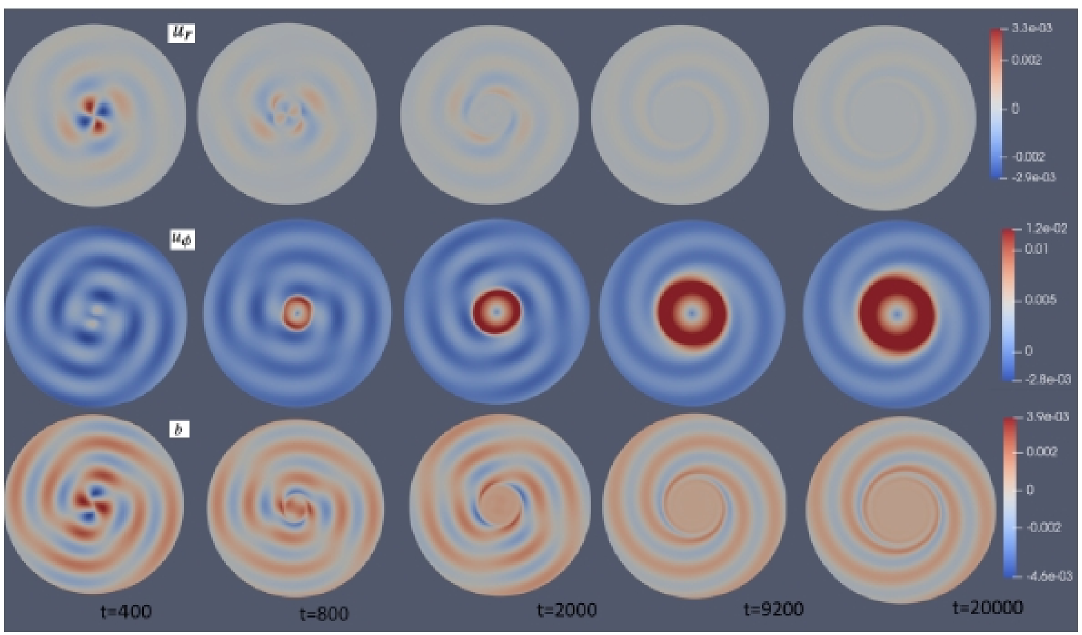


Figure 16. u_r, u_ϕ (radial and azimuthal velocities) and b (buoyancy) for a simulation with high-amplitude forcing ($U = 10^{-4}$) with frequency $\omega = 0.11$. The diffusion coefficients are $\nu = 10^{-6}$ and $\kappa = 5 \times 10^{-6}$.

Dynamical picture (Guo, Ogilvie, Barker, 2023)

Conclusions

For slowly rotating MS stars the linear theory of dynamic tides associated with g-modes is reasonably well understood. For certrain astronomical systems (but, not all of them!), when tides are assumed to operate in the regime of moderately large viscosity, the theory gives a good agreement with observations.

But, there are certain issues with so-called inertial waves – another branch of perturbations of potential importance. However, a lot of work has already been done to clarify the situation.

Non-linear effects are imporant to justify the regime of moderately large viscosity. They can also, in principle, explain a disargeement between the theory and observations in certain cases. So far, only some simple estimates or model numerical calculations have been done in this field. Clearly, we need much more work to be done in this direction.