Slipping flows and their breaking

E.A. Kuznetsov [1,2,3] and E.A. Mikhailov[1,3,4]

[1] - P.N. Lebedev Physical Institute, Moscow, Russia

[2] - L.D. Landau Institute for Theoretical Physics, Chernogolovka, Moscow region, Russia

[3] - Skoltech, Skolkovo, Moscow region, Russia

[4] - Physics Dept, Moscow State University

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OUTLINE

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- Solution to the inviscid Prandtl equation
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- According to the Kolmogorov-Obukhov theory (1941) velocity fluctuations at spatial scales l from the inertial range obey the power-law $\langle|\delta v|\rangle \propto \varepsilon^{1/3} l^{1/3}$, where ε is the mean energy flux from large to small scales. This formula is easily obtained from the dimensional analysis.
- Similarly, fluctuations for the vorticity field $\omega = \nabla \times \mathbf{v}$ diverge at small scales as $\langle|\delta\omega|\rangle \propto \varepsilon^{1/3} l^{-2/3}$, while the time of energy transfer from the energy-contained scale l_E to the viscous ones is finite and estimated as $T \sim l_E^{2/3} \varepsilon^{-1/3}$.
- These two relations allow to link the Kolmogorov spectrum formation with the blowup in the vorticity field (collapse).

Introduction & Motivation

- The question whether finite time singularities develop in inertial scales (in fact, in ideal fluids) is still open question, in spite of certain progress in both numerical and analytical studies.
- Up to now, the question about blow-up existence for ideal fluids within the 3D Euler remains controversial. In our numerics (Agafontsev, Kuznetsov, Mailybaev 2015, 2017, 2019, 2022) for periodical boxes we have observed formation of high-vorticity structures of the pancake type with exponential growth of ω but without any tendency to blowup. Such increasing is connected with the vorticity compressibility. The latter follows from the vorticity ω frozen-in-fluids.

Introduction & Motivation

- However, for flows of ideal fluids in the presence of rigid boundaries recent findings, both analytical and numerical, demonstrate blow-up behavior. For two-dimensional planar flows in the region with non-smooth boundaries Kiselev and Zlatos (2015) proved blow-up existence.
- In 2014, 2015 Luo and Hou in numerical experiments for axi-symmetrical flows with swirl inside the cylinder of constant radius observed appearance of collapse just on the boundary. It was a challenge why boundaries play so important role in formation of singularities.
- In 2019 Elgindi and Jeong proved the existence of solutions to the axi-symmetric 3D Euler equations outside the cylinder $(1+\epsilon|z|)^2 \le x^2+y^2$ with singularity on the wall.

Introduction & Motivation

- The latter result correlates with studies of Kiselev and Zlatos (2015) for two-dimensional Euler flow inside the region with not-smooth boundaries.
- In 1985 E and Engquist reported some rigorous results about blow-up existence for both inviscid and viscous Prandtl equation for some initial data when the velocity component parallel to a wall vanishes at the whole vertical line. For such initial conditions these authors found sufficient condition for blow-up in the viscous case.
- It is worth noting that before, in 1980, the blowup appearance in the Prandtl equation was observed in the numerical simulations by Van Dommelen and Shen.

Introduction & Motivation

- In 2003 Hong and Hunter investigated this problem for both viscous and inviscid Prandtl equation for zeroth pressure gradient. In particular, in the inviscid case they noticed that singularity can form on the wall.
- In 2014 for smooth boundary conditions in the case of 2D Euler for flows inside a disk Kiselev and Šverák constructed an initial data for which the gradient of vorticity exhibits double exponential growth in time with maximum value on the boundary. Simultaneously the velocity gradient grows on the boundary exponentially in time.

Introduction & Motivation

- In this lecture we show that flat boundary itself introduces some element of compressibility into flow which from our point of view can be considered as a reason of the singularity formation on the boundary. We will consider the 2D and 3D inviscid Prandtl equations which describes the dynamics of the boundary layer, and demonstrate that singularity is formed for the velocity gradient on the wall. For 2D Euler numerically we show that for flows between two parallel plates the maximal velocity gradient grows exponentially in time on the wall and the vorticity gradient has a tendency for double exponential growth there. This process is nothing more than breaking (or folding for 2D Euler) phenomenon which is well known in gas dynamics since the classical works of famous Riemann.

Basic equations and mixed Lagrangian-Eulerian description

The inviscid Prandtl equation for 2D flows is written for the velocity component parallel to the blowing plane $y = 0$:

$$u_t + uu_x + vu_y = -P_x, \; u_x + v_y = 0$$

with the following initial-boundary conditions:
$u(x, y, 0) = u_0(x, y), \; v(x, y, 0) = v_0(x, y)$, and
$v|_{y=0} = 0, \; \lim_{y\to\infty} u(x, y, t) = U(x).$
In terms of stream function ψ ($u = \psi_y$, $v = -\psi_x$) these conditions read as $\psi(x, y, 0) = \psi_0(x, y), \psi(x, 0, t) = 0,$ and at $y \to \infty \; \psi(x, y, t) \to U(x)y.$
Here pressure P is independent on both y and t and satisfying the Bernoulli law:

$$\frac{U^2(x)}{2} + P(x) = \text{const.}$$

Basic equations and mixed Lagrangian-Eulerian description

NOTE: The Prandtl equation assumes that the along surface scale L much larger the boundary layer thickness h: $L \gg h$. Hence one can see from incompressibility condition that $u/L \approx v/h$, i.e. $u \gg v$. As a result, the pressure $P = P(x)$. It gives the Prandtl equations

$$u_t + uu_x + vu_y = -P_x, \; u_x + v_y = 0$$

Basic equations and mixed Lagrangian-Eulerian description

Within the Prandtl approximation for inviscid flows it is possible to introduce the vorticity as

$$\omega = -\frac{\partial u}{\partial y}$$

which satisfies the equation of the same form as for the 2D Euler fluids:

$$\omega_t + u\omega_x + v\omega_y = 0.$$

Thus, ω is the Lagrangian invariant. By this reason, its values will be bounded at all $t > 0$. However, for another components of the velocity gradient such restrictions are absent. As we will see below, u_x as well as v_y can take arbitrary values, in particular, infinite ones.

Basic equations and mixed Lagrangian-Eulerian description

For $P_x = 0$, u is a Lagrangian invariant. Let n be some Lagrangian quantity (advected by the fluid), obeys the equation

$$n_t + un_x + vn_y = 0, \ \ u_x + v_y = 0.$$

For its solution $n = n(x, y, t)$, define inverse function $y = y(x, n, t)$. In this case we have new independent Lagrangian variable n and old Eulerian coordinate x (note, for the Prandtl equation such transformation was introduced first time by Crocco). Transition to this description is the mixed Eulerian-Lagrangian one which represents non-complete Legendre transformation. Fixing n in $y = y(n, x, t)$ yields the n-level line and therefore this transform is the transition to the movable *curvilinear* system of coordinates.

Basic equations and mixed Lagrangian-Eulerian description

Then we find how derivatives with respect to variables (x, y, t) (the l.h.s) and derivatives relative to (x, n, t) (the r.h.s) are connected with each other:

$$\frac{\partial f}{\partial t} = \frac{1}{y_n}[f_t y_n - f_n y_t],$$
$$\frac{\partial f}{\partial x} = \frac{1}{y_n}[f_x y_n - f_n y_x],$$
$$\frac{\partial f}{\partial y} = \frac{f_n}{y_n}.$$

Substitution of these transforms into the equation of motion for n gives the kinematic condition, well known for free-surface hydrodynamics:

$$y_t + u y_x = v.$$

Basic equations and mixed Lagrangian-Eulerian description

Introduce streamfunction ψ so that $u = \psi_y, \ v = -\psi_x$. By means of formulas for derivatives these relations read as

$$u = \frac{1}{y_n}\psi_n, \ v = -\psi_x + \frac{y_x}{y_n}\psi_n.$$

Substitution of these formulas into the equation for y results in the linear relation between y and ψ:

$$y_t = -\psi_x.$$

Note, that in this equation all derivatives are taken for fixed n. This equation can be easily resolved by introducing the generating function $\theta(x, n, t)$:

$$y = \theta_x, \ \psi = -\theta_t.$$

To find $\theta(x, n, t)$ one needs to know dynamics of the velocity.

Solution to the inviscid Prandtl equation

Consider first $P_x = 0$. In this case for the inviscid Prandtl equation we have Eq.

$$u = \frac{1}{y_u}\,\psi_u,$$

which after substitution of θ transforms into

$$\frac{\partial \theta_u}{\partial t} + u\,\frac{\partial \theta_u}{\partial x} = 0.$$

This equation evidently has the following solution:

$$\theta_u = F(x - ut, u)$$

where F is an arbitrary smooth function determined from the boundary-initial conditions. Integration with respect to u yields

$$\theta = \int_{f(x,t)}^{u} F(x - zt, z)dz + g(x,t).$$

Solution to the inviscid Prandtl equation

Here $f(x,t)$ and $g(x,t)$ are another arbitrary functions to be defined from the B-I conditions.

It is worth noting that at $y=0$ and $P_x=0$ the inviscid Prandtl equation is nothing more than the Hopf equation

$$u_t + uu_x = 0,$$

which solution is written in the following implicit form (simple Riemann wave)

$$u = u_0(a), \;\; x = a + u_0(a)t$$

or

$$u = u_0(x - ut).$$

This means that on the boundary we have breaking, i.e. the formation of singularity in a finite time.

Solution to the inviscid Prandtl equation

Breaking happens when the derivative

$$\frac{\partial u}{\partial x} = \frac{u_0'(a)}{1 + u_0'(a)t}$$

at some point $x = x_*$ first time, $t = t_*$, becomes infinite. It is evident that $t_* = \min_a \left[-1/u_0'(a)\right].$ Then it is possible to establish that the general solution is matched with the boundary conditions at $y = 0$ if one puts

$$f(x,t) = u(x,0,t)$$

(this is solution of the Hopf equation) and $g(x,t) = 0$ so that

$$\begin{aligned} y &= \int_{f(x,t)}^{u} \frac{\partial}{\partial x} F[x - zt, z] dz \\ \psi &= -\int_{f(x,t)}^{u} \frac{\partial}{\partial t} F[x - zt, z] dz. \end{aligned}$$

Solution to the inviscid Prandtl equation

Near the breaking point,

$$u_x \simeq -\frac{1}{\tau + \beta(\Delta a)^2}$$

where $\tau = t_* - t,\ \Delta a = a - a_*$.

Thus, this dependence demonstrates a self-similar compression, $\Delta a \propto \tau^{1/2}$. The denominator, up to the constant multiplier C, coincides with the Jacobian, $J = \partial x/\partial a = C\left(\tau + \beta a^2\right),$ where we put $a_* = 0$. Integration of this equation gives the cubic dependence: $x = C\left(\tau a + \beta a^3/3\right)$. Thus, in the physical space we get more rapid compression than in the a-space : $x \propto \tau^{3/2}$.

Solution to the inviscid Prandtl equation

For $\beta a^2 \gg \tau$, the Jacobian becomes time-independent, $J \sim x^{2/3}$. Hence, as $\tau \to 0$ we arrive at the singularity for u_x, $u_x \propto x^{-2/3}$. Any time changes of u_x happen at the narrowing region in the a-space, $a \propto \tau^{1/2}$, or equivalently at $x \propto \tau^{3/2}$. It results in the following self-similar asymptotics,

$$u_x \simeq \frac{1}{\tau} F(\xi), \ \xi = \frac{x}{\tau^{3/2}}$$

where function $F(\xi)$ as $\xi \to \infty$ is $\sim \xi^{-2/3}$. Hence we have the power law:

$$\max |u_x| \propto \ell^{-2/3}.$$

This is a general asymptotics for folding, independently whether the singularity happens in finite or infinite time.

Solution to the inviscid Prandtl equation

For arbitrary dependence $P(x)$ a solution is found from integration of ODEs:

$$\frac{d}{dt}u = -P_x, \ \frac{d}{dt}x = u,$$

which are equivalent to the Newton equation: $\ddot{x} = -P_x$. The first integral (energy) $E(a) = \dot{x}^2/2 + P(x) = u_0^2(a)/2 + P(a)$, allows to define the mapping $x = x(a,t)$. The breaking time t_* is found as a minimal root $T(> 0)$ for equation $J(a,T) = 0$, where $t_* = \min_a T(a)$ and $J = \partial x/\partial a$.

Behavior for the vorticity gradients on the boundary

Now calculate how ω behaves at the breaking point. Remind, $\omega = -u_y$ is the Lagrangian invariant.

For simplicity consider the pressureless case. Differentiation of the vorticity equation with respect to x and then putting $y = 0$ where $v = 0$ and $v_x = 0$ yield the following

$$\frac{\partial \omega_x}{\partial t} + u \frac{\partial}{\partial x} \omega_x = -u_x \omega_x .$$

The equations for characteristics are $dx/dt = u(x,t), \ \ d\omega_x/dt = -u_x \omega_x$. Substitution of $u_x \simeq (t - t_*)^{-1}$ at the breaking point gives the same singular behavior for ω_x there:

$$\omega_x \simeq \frac{A}{t - t_*} .$$

Solution to the 2D inviscid Prandtl equation

Concluding this part, note that singularities for the velocity gradient on the boundary is a result of collision of two counter-propagating slipping flows. In the first simulations (Dommelen and Shen, 1980; Hong and Hunter, 2003) this interaction was shown to lead to the formation of jets in perpendicular to the boundary direction. Breaking (as a folding happening in a finite time) for the slipping flows in the 2D Prandtl equation becomes possible because the pressure gradient normal to the boundary can not prevent the formation of jets.

Folding of the slipping flows for the 2D Euler equation

Now consider slipping flows within the 2D Euler equation between two parallel plates and present numerical results for folding of such flows which develops on the plate boundary. As shown by Kiselev and Šverák (2014) for the 2D Euler flows inside a disk for some initial data the gradient of ω exhibits double exponential growth in time with maximum value on the boundary. Our numerical results are in the correspondence with this paper. In particular, we have observed that $\max|u_x|$ at the wall grows in time approximately exponentially like for a disk. This results in the double exponential growth of the vorticity gradient for the 2D Euler flows. This process can be considered as folding with typical dependence between growing $\max|u_x|$ and its narrowing in time width ℓ: $\max|u_x| \propto \ell^{-2/3}$.

Folding of the slipping flows for the 2D Euler equation

Numerically we solve the 2D Euler equation for ω

$$\omega_t + u\omega_x + v\omega_y = 0$$

for flows between two rigid plates $y = 0$ and $y = h$, with slip boundary conditions (BC), $v(x,0) = v(x,h) = 0$, and periodical BC along x. The velocity components u and v are found through the streamfunction ψ. For ψ we use the **zero** BC at $y = 0$ and $y = h$. Such choice corresponds to the absence of the flow with a constant velocity along x-direction. For integration of the equations we used the pseudo-transient method and Peaceman-Rachford scheme. The accuracy for the first one was $||\Delta\psi + \omega||^2 \leq 10^{-7}$. The kinetic energy in our simulations was conserved not worth than 10^{-6}.

Folding of the slipping flows for the 2D Euler equation

Now we present results of our simulations for the following initial conditions (IC):
$\psi(x,y,0) = -By(y-h)^2 \sin x;\ h = 2,\ B = 0.1.$ These IC were chosen so that the folding point appears at $x = 0$ on the boundary $y = 0$.
At first, the spatiotemporal dependencies of velocity were found numerically and then the temporal evolution of its gradient was determined. Analysis of the distribution of the velocity gradient showed that for the IC almost from the very beginning its maximum is concentrated on the boundary $y = 0$ in the vicinity of the point $x = 0$ which corresponds to the folding point. Around this point the parallel velocity u behaves almost like for overturning describing by the Hopf equation.

Folding of the slipping flows for the 2D Euler equation

Black line corresponds to $t = 0$, red – $t = 1$, blue – $t = 2$, green – $t = 5$.

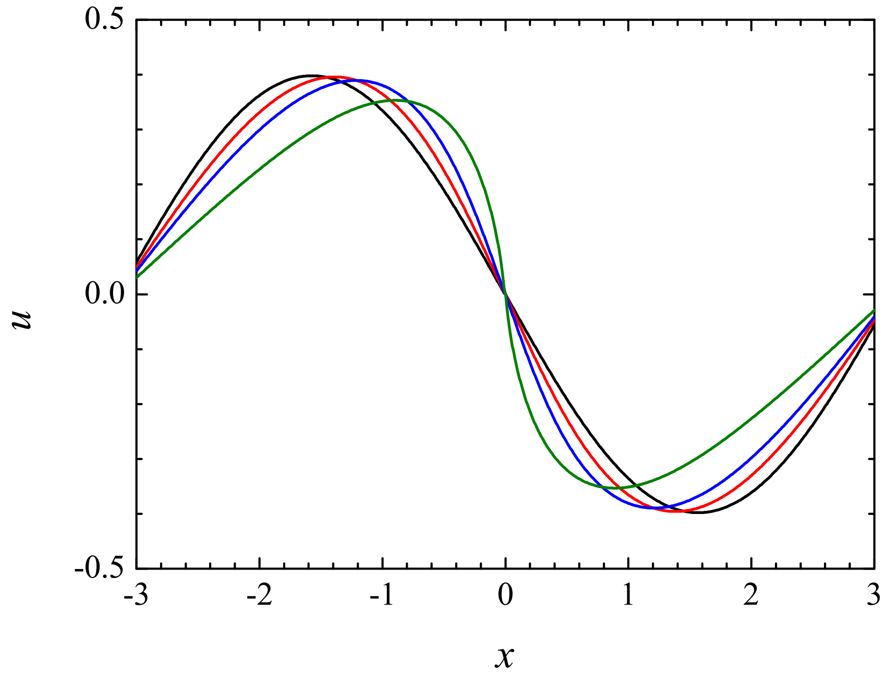

Figure 1: Dependencies of the slipping ($y = 0$) velocity as a function of x at different moments of time.

Folding of the slipping flows for the 2D Euler equation

With time increasing u_x is seen to transform into a cusp. u_x becomes more and more negative.

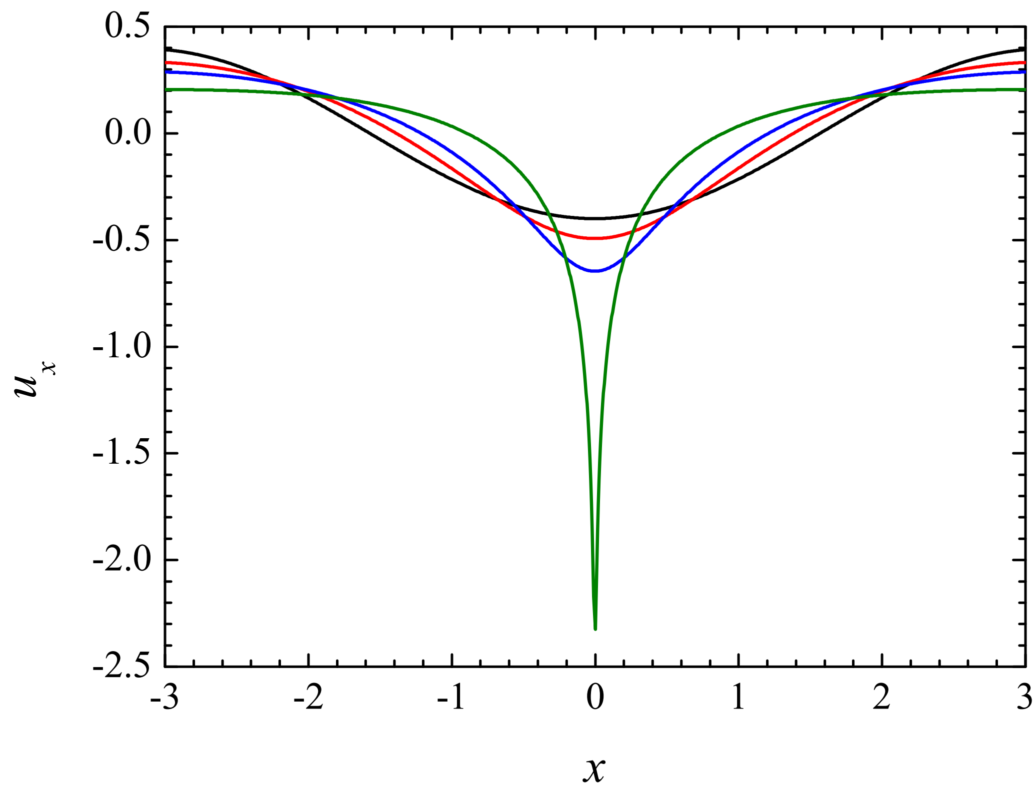

Figure 2: Dependencies of the slipping u_x as a function of x at different times.

Folding of the slipping flows for the 2D Euler equation

This figure demonstrates the exponential growth for maximum $|u_x|$.The blacks are the numerical results, the red is $\propto e^{\gamma_1 t}$ with $\gamma_1 = 0.44$.

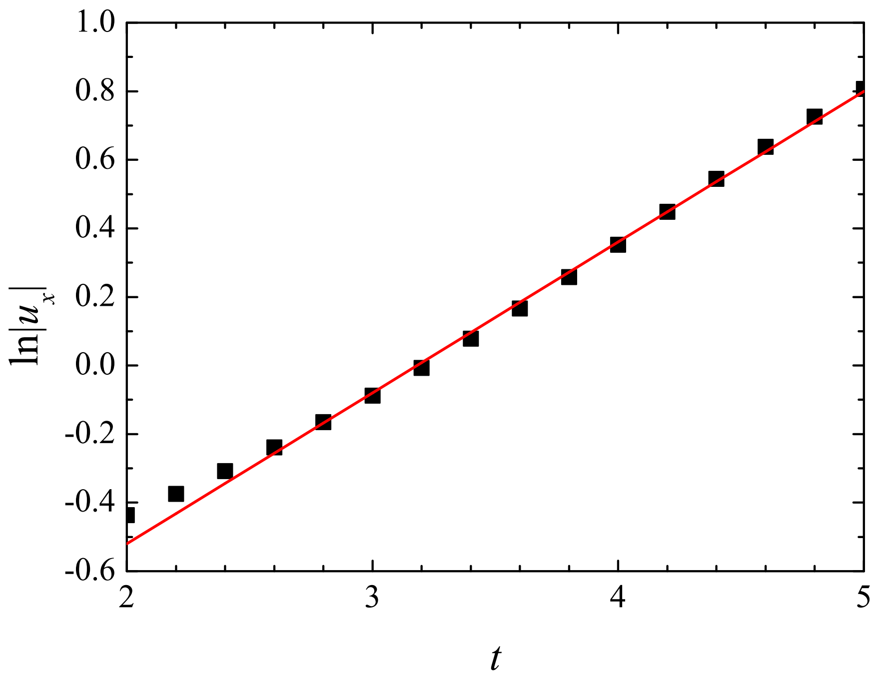

Figure 3: Time dependence of $\max |u_x|$ for the slipping flow (logarithmic scale).

Folding of the slipping flows for the 2D Euler equation

The thickness ℓ shows an exponential decrease.

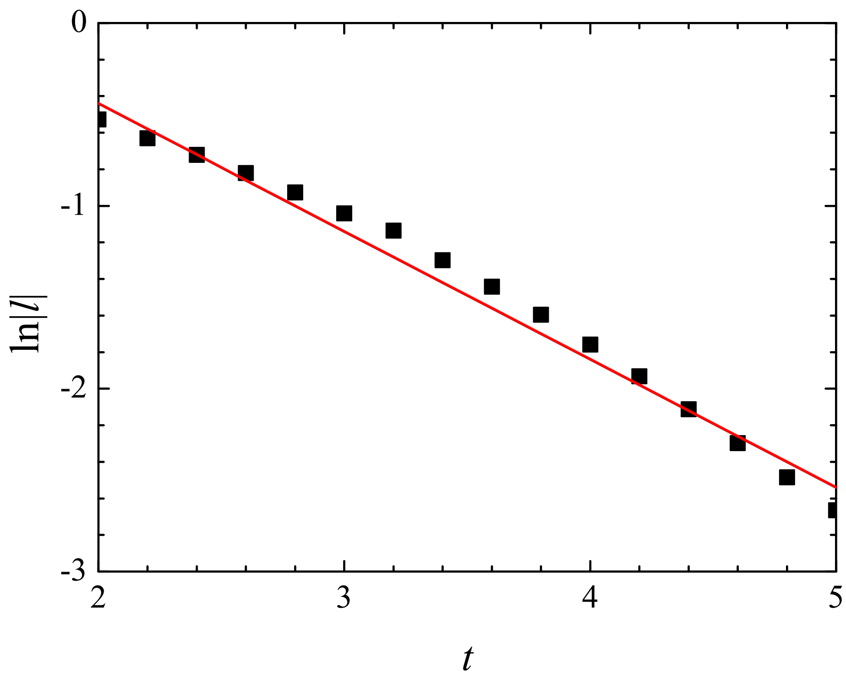

Figure 4: Spatial thickness of $|u_x|$ for the slipping ($y = 0$) flow as a function of time. Blacks are numerical results, red is the slope $\propto e^{-\gamma_2 t}$ with $\gamma_2 = 0.7$.

Folding of the slipping flows for the 2D Euler equation

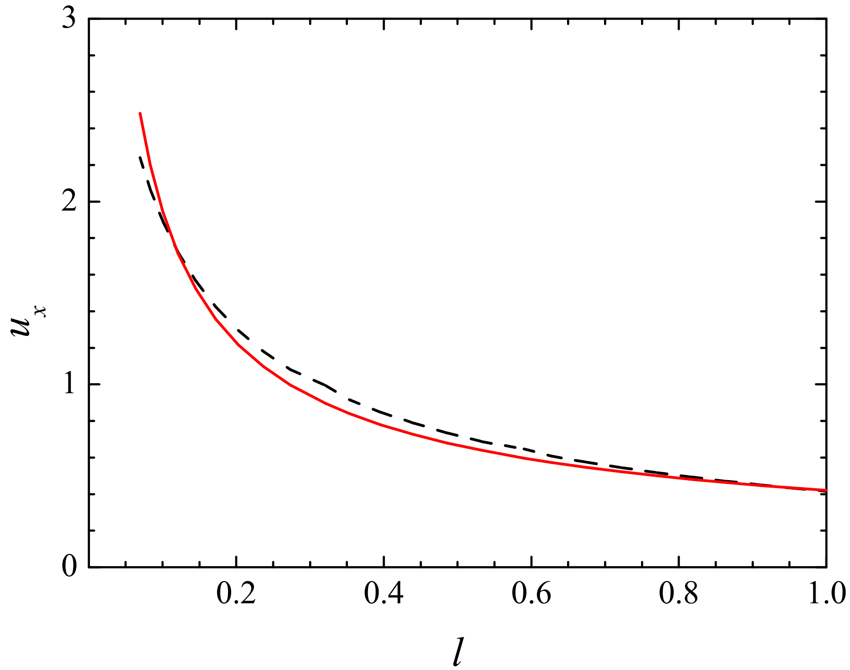

Figure 5: $\max|u_x|$ versus thickness ℓ. Black dashed line is numerics, and red is the power dependence $\max|u_x| \propto \ell^{-2/3}$.

Folding of the slipping flows for the 2D Euler equation

Such behavior means that the power law dependence arises between $\max|u_x|$ and ℓ,

$$\max|u_x| \propto \ell^{-\alpha},$$

with exponent $\alpha \approx 2/3$. Thus, this process for the slipping flow can be considered as a folding.

Folding of the slipping flows for the 2D Euler equation

The folding results in the formation of jet in transverse direction to the boundary $y = 0$.

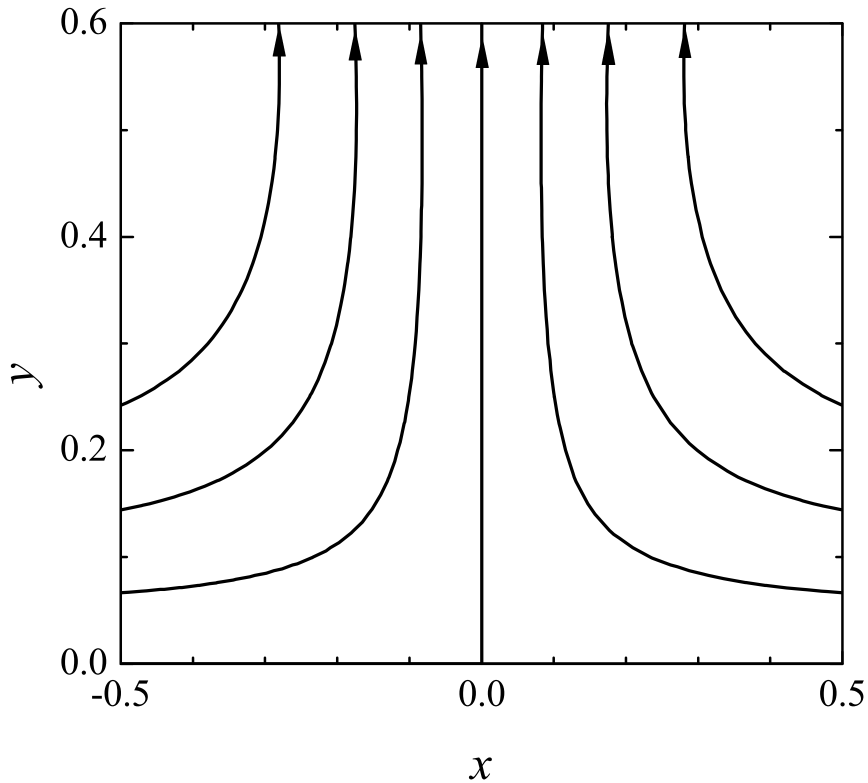

Figure 6: Streamlines at $t = 5$ in the neighborhood of $x = y = 0$.

Folding of the slipping flows for the 2D Euler equation

This Fig. shows the process of the jet formation for the streamfunction levels $\psi = \pm 0.01$ (plus corresponds to negative x, minus to $x > 0$). The fixed difference $\Delta\psi = 0.02$ means that the fluid flux between lines $\psi = \pm 0.01$ remains constant, but the region between them with time becomes more narrow that corresponds to the velocity increase in perpendicular direction relative to the slipping flow.

Folding of the slipping flows for the 2D Euler equation

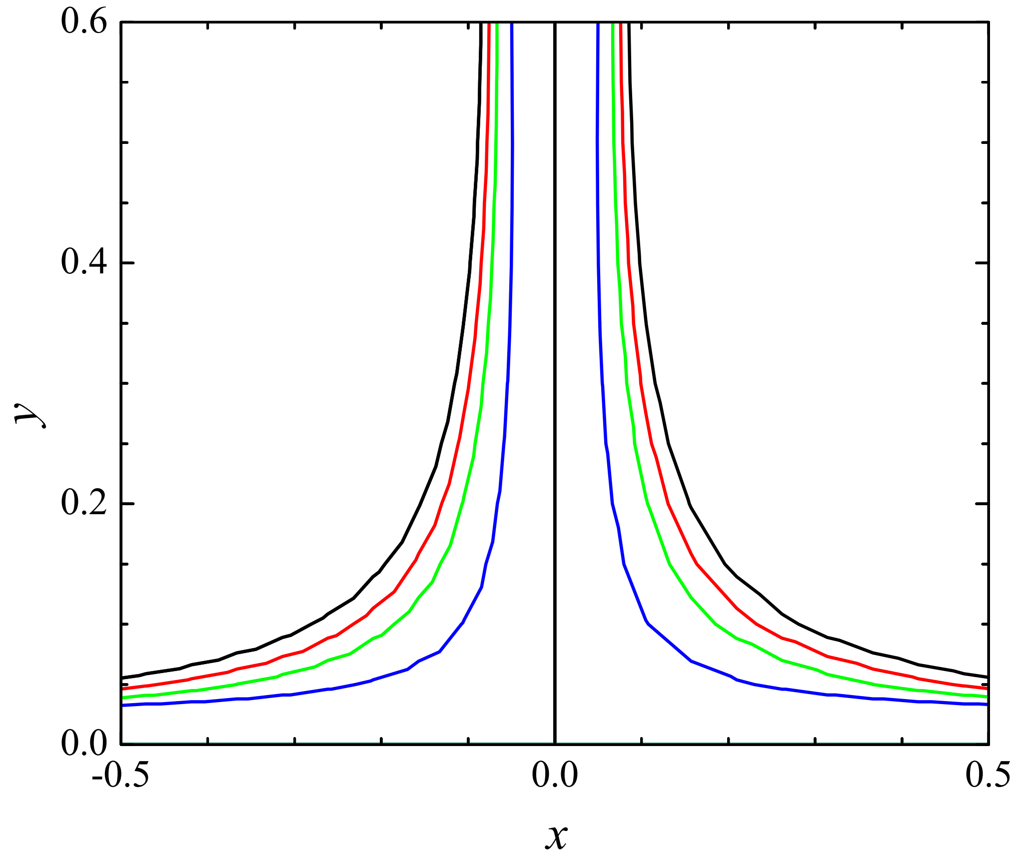

Figure 7: Behavior of streamfunction levels $\psi = \pm 0.01$ at different times.

Folding of the slipping flows for the 2D Euler equation

The physical reason of the jet appearance is connected with collision of two counter-propagating slipping flows. Exponential growth of $|u_x|$ should promote the vorticity gradient increase during the folding process. It follows from the equation for di-vorticity at $y = 0$:

$$\frac{\partial \omega_x}{\partial t} + u\frac{\partial}{\partial x}\omega_x = -u_x\omega_x.$$

This equation can be solved by the method of characteristics: $dx/dt = u(x,t),\ d\omega_x/dt = -u_x\omega_x$. From the second equation we get double exponential growth for ω_x if increase of u_x is exponential:

$$\log \omega_x = -\int^t u_x dt'.$$

Folding of the slipping flows for the 2D Euler equation

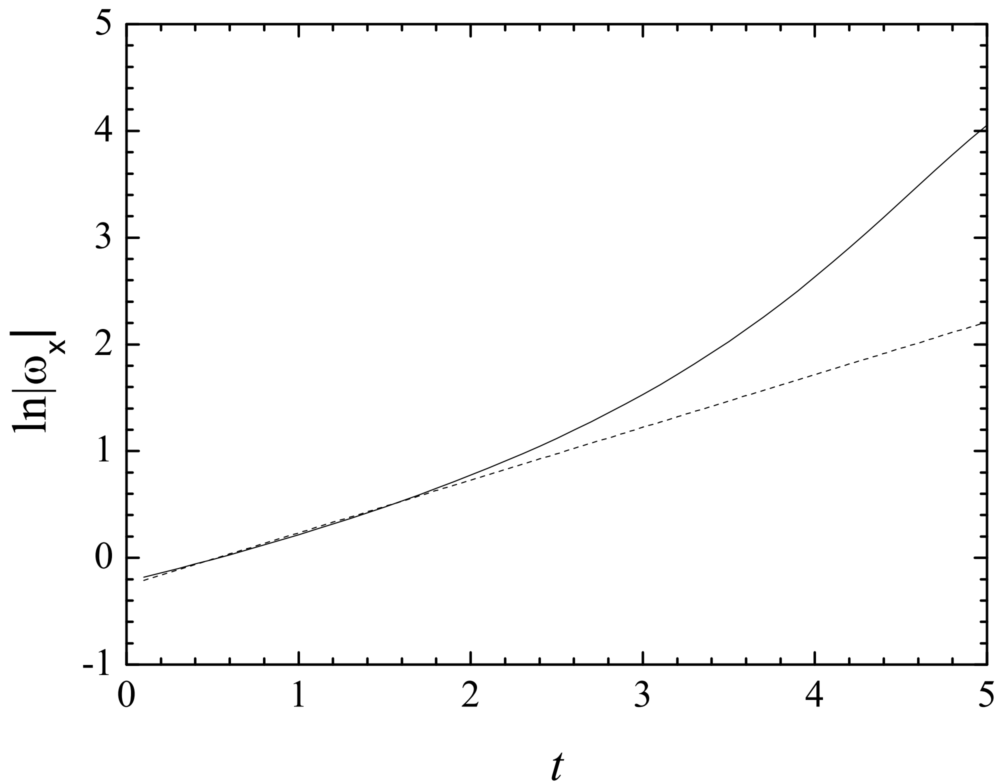

Figure 8: Dependence of ω_x at $x = 0$ (logarithmic scale) versus time. Dashed line shows exponential behavior at initial times.

Folding of the slipping flows for the 2D Euler equation

As already noted, at the folding region $u_x < 0$. If $\max |u_x|$ increases exponentially in time for the slipping flow then ω_x will have a double exponential growth in time. Our numerical simulations support this conclusion. In the logarithmic scales as it is seen from Fig. 8 initially the growth of $\ln \omega_x$ at the folding point $x = 0$ looks like exponential straight line) but at the later stage one can see positive deviation from this line. ω_x grows exponentially faster that is in accordance with the theoretical arguments. From another side, the fitting for dependence $\max |u_x|$ indicates its exponential in time increase. If it is so then Fig. 8 can be considered as a certain conformation in the favor to existence of the double exponential growth of ω_x. Thus, our numerical results correspond to those by Kiselev and Šverák for the Eulerian flow inside a disk.

Breaking in the 3D inviscid Prandtl equations

- The 3D inviscid Prandtl equations have the form

$$\mathbf{u}_t + (\mathbf{u}\nabla)\mathbf{u} + v\mathbf{u}_z = -\nabla P(\mathbf{r}), \ \ (\nabla \mathbf{u}) + v_z = 0$$

where $\mathbf{r} = (x, y)$ and $\mathbf{u}$ are, respectively, coordinates and velocity components parallel to the wall, $\nabla = (\partial_x, \partial_y)$, v is the normal ($\| z$) velocity component.

- Hence for slipping boundary conditions we arrive at the 2D Hopf equation

$$\mathbf{u}_t + (\mathbf{u}\nabla)\mathbf{u} = -\nabla P(\mathbf{r})$$

which also gives breaking.

Breaking in the 3D inviscid Prandtl equations

Consider for simplicity the case $P(\mathbf{r}) = \text{const}$. Then the velocity gradient $U_{ij} = \partial u_i / \partial x_j$ satisfies the following matrix equation

$$\frac{dU}{dt} = -U^2$$

which solution has the form

$$U = U_0(a)(1 + U_0(a)t)^{-1}$$

where $U_0(a)$ and a are the initial values of U and positions of fluid particles.

Breaking in the 3D inviscid Prandtl equations

Expanding $U_0(a)$ through the projectors P_α yields

$$U = \sum_\alpha \frac{\lambda_\alpha P_\alpha}{1 + \lambda_\alpha t}$$

Hence it is seen that the breaking time

$$t_0 = \min_{\alpha,a} (-\lambda_\alpha)^{-1}.$$

Near $t = t_0$

$$U \propto (t_0 - t)^{-1}$$

with the main contribution originating from the eigen value corresponding to t_0.

Breaking in the 3D inviscid Prandtl equations

This gives simultaneous singularities for both symmetric part (stress tensor)

$$S = 1/2(U + U^T)$$

and antisymmetric part (vorticity)

$$\Omega = U - U^T.$$

Singularities for both parts have the cusp form, like in the 1D case. Note that in this case, unlike 1D where the breaking criterion is written as $u_0' < 0$, we have a few restrictions on λ which are defined from quadratic equation. The first condition is that eigen values λ should be real. Secondly, λ has to take negative values.

Breaking in the 3D inviscid Prandtl equations

As we see breaking of the slipping flows in 2D Prandtl and 2D Euler is accompanied by the appearance of jets in the perpendicular direction to the slipping boundary. The same situation takes place in 3D (at least, for the inviscid Prandtl equations). From another side, breaking (or folding happening in a finite time) in the inviscid Prandtl approximation for the general initial conditions should produce growth of the perpendicular to the slipping boundary vorticity. Combination of these both factors gives an indication for understanding a mechanism of tornado generation.

Conclusion

- We have developed a new concept of the formation of big velocity gradients with the blow-up behavior or with the exponential in time increase for the slipping flows in incompressible inviscid fluids. These processes develop as a folding due to compressible character of the slipping flows.
- For the 2D inviscid Prandtl equation we have developed the mixed Lagrangian – Eulerian description based on the Crocco transformation.
- Application of this description to the inviscid Prandtl equation allows to construct its general solution written in the implicit form.

Conclusion

- It has been demonstrated that for the inviscid Prandtl equation appearance of the finite-time singularity can happen on the wall.
- For 2D Euler flows we have numerically found that maximum of the velocity gradient is developed on the plate with exponential increase in time. Simultaneously, the vorticity gradient has been shown to demonstrate the double exponential growth in time on the boundary.

THANKS FOR YOUR ATTENTION